INFINITE DETERMINANTAL MEASURES AND THE ERGODIC DECOMPOSITION OF INFINITE PICKRELL MEASURES

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ABSTRACT. The main result of this paper, Theorem 1.11, gives an explicit description of the ergodic decomposition for infinite Pickrell measures on spaces of infinite complex matrices. The main construction is that of sigma-finite analogues of determinantal measures on spaces of configurations. An example is the infinite Bessel point process, the scaling limit of sigma-finite analogues of Jacobi orthogonal polynomial ensembles. The statement of Theorem 1.11 is that the infinite Bessel point process (subject to an appropriate change of variables) is precisely the ergodic decomposition measure for infinite Pickrell measures.

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1. INTRODUCTION

1.1. **Informal outline of the main results.** The Pickrell family of mesures is given by the formula

$$\mu_n^{(s)} = \mathrm{const}_{n,s} \det(1 + z^* z)^{-2n-s} dz. \tag{1}$$

Here n is a natural number, s a real number, z a square $n \times n$ matrix with complex entries, dz the Lebesgue measure on the space of such matrices, and $\mathrm{const}_{n,s}$ a normalization constant whose precise choice will be explained later. The measure $\mu_n^{(s)}$ is finite if $s > -1$ and infinite if $s \leq -1$. By definition, the measure $\mu_n^{(s)}$ is invariant under the actions of the unitary group $U(n)$ by multiplication on the left and on the right.

If the constants $\mathrm{const}_{n,s}$ are chosen appropriately, then the Pickrell family of measures has the Kolmogorov property of consistency under natural projections: the push-forward of the Pickrell measure $\mu_{n+1}^{(s)}$ under the natural projection of cuttting the $n \times n$-corner of a $(n+1) \times (n+1)$-matrix is precisely the Pickrell measure $\mu_n^{(s)}$. This consistency property is also verified for infinite Pickrell measures provided n is sufficiently large; see Proposition 1.8 for the precise formulation. The consistency property and the Kolmogorov Theorem allows one to define the Pickrell family of measures $\mu^{(s)}$, $s \in \mathbb{R}$, on the space of infinite complex matrices. The Pickrell measures are invariant by the action of the infinite unitary group on the left and on the right, and the Pickrell family of measures is the natural analogue, in infinite dimension, of the canonical unitarily-invariant measure on a Grassmann manifold, see Pickrell [33].

What is the ergodic decomposition of Pickrell measures with respect to the action of the Cartesian square of the infinite unitary group? The ergodic unitarily-invariant probability measures on the space of infinite complex matrices admit an explicit classification due to Pickrell [31] and to which Olshanski and Vershik [30] gave a different approach: each ergodic measure is determined by an infinite array $x = (x_1, \dots, x_n, \dots)$ on the half-line, satisfying $x_1 \geq x_2 \cdots \geq 0$ and $x_1 + \dots + x_n + \dots < +\infty$, and an additional parameter $\tilde{\gamma}$ that we call the *Gaussian parameter*. Informally, the parameters x_n should be thought of as "asymptotic singular values" of an infinite complex matrix, while $\tilde{\gamma}$ is the difference between the "asymptotic trace" and the sum of asymptotic eigenvalues (this difference is positive, in particular, for a Gaussian measure).

Borodin and Olshanski [6] proved in 2000 that for finite Pickrell mesures the Gaussian parameter vanishes almost surely, and the ergodic decomposition measure, considered as a measure on the space of conifgurations on the half-line $(0, +\infty)$, coincides with the Bessel point process of Tracy and Widom [43], whose correlation functions are given as determinants of the Bessel kernel.

Borodin and Olshanski [6] posed the problem: *Describe the ergodic decomposition of infinite Pickrell measures.* This paper gives a solution to the problem of Borodin and Olshanski.

The first step is the result of [10] that almost all ergodic components of an infinite Pickrell measure are themselves *finite*: only the decomposition measure itself is infinite. Furthermore, it will develop that, just as for finite measures, the Gaussian parameter vanishes. The ergodic decomposition measure can thus be identified with a sigma-finite measure $\mathbb{B}^{(s)}$ on the space of configurations on the half-line $(0, +\infty)$.

How to describe a sigma-finite measure on the space of configurations? Note that the formalism of correlation functions is completely unapplicable, since these can only be defined for a finite measure.

This paper gives, for the first time, an explicit method for constructing infinite measures on spaces of configurations; since these measures are very closely related to determinantal probability measures, they are called *infinite determinantal measures*.

We give three descriptions of the measure $\mathbb{B}^{(s)}$; the first two can be carried out in much greater generality.

- *Inductive limit of determinantal measures.* By definition, the measure $\mathbb{B}^{(s)}$ is supported on the set of configurations X whose particles only accumulate to zero, not to infinity. $\mathbb{B}^{(s)}$-almost every configuration X thus admits a maximal particle $x_{max}(X)$. Now, if one takes an arbitrary $R > 0$ and restricts the measure $\mathbb{B}^{(s)}$ onto the set $\{X : x_{max}(X) < R\}$, then the resulting restricted measure is finite and, after normalization, determinantal. The corresponding operator is an orthogonal projection operator whose range is found explicitly for any $R > 0$. The measure $\mathbb{B}^{(s)}$ is thus obtained as an inductive limit of finite determinantal measures along an exhausting family of subsets of the space of configurations.

- *A determinantal measure times a multiplicative functional.* More generally, one reduces the measure $\mathbb{B}^{(s)}$ to a finite determinantal measure by taking the product with a suitable multiplicative functional. A *multiplicative functional* on the space of configurations is obtained by taking the product of the values of a fixed nonnegative function over all particles of a configuration:

$$\Psi_g(X) = \prod_{x \in X} g(x).$$

If g is suitably chosen, then the measure

$$\Psi_g \mathbb{B}^{(s)} \tag{2}$$

is finite and, after normalization, determinantal. The corresponding operator is an orthogonal projection operator whose range is found explicitly. Of course, the previous description is a particular case of this one with $g = \chi_{(0,R)}$. It is often convenient to take a positive function, for example, the function $g^\beta(x) = \exp(-\beta x)$ for $\beta > 0$. While the range of the orthogonal projection operator inducing the measure (2) is found explicitly for a wide class of functions g, it seems possible to give a formula for its kernel for only very few functions; these computations will appear in the sequel to this paper.

- *A skew-product.* As was noted above, $\mathbb{B}^{(s)}$-almost every configuration X admits a maximal particle $x_{max}(X)$, and it is natural to consider conditional measures of the measure $\mathbb{B}^{(s)}$ with respect to the position of the maximal particle $x_{max}(X)$. One obtains a well-defined determinantal probability measure induced by a projection operator whose range, again, is found explicitly using the description of Palm measures of determinantal point processes due to Shirai and Takahashi [39] . The sigma-finite distribution of the maximal particle is also explicitly found: the ratios of the measures of intervals are obtained as ratios of suitable Fredholm determinants. The measure $\mathbb{B}^{(s)}$ is thus represented as a skew-product whose base is an explicitly found sigma-finite measure on the half-line, and whose fibres are explicitly found determinantal probability measures. See section 1.10 for a detailed presentation.

The key rôle in the construction of infinite determinantal measures is played by the result of [11] (see also [12]) that a determinantal probability measure times an integrable multiplicative functional is, after normalization, again a determinantal probability measure whose operator is found explicitly. In particular, if $\mathbb{P}_\Pi$ is a determinantal point process induced by a projection operator Π with range L, then, under certain additional assumptions, the measure $\Psi_g\mathbb{P}_\Pi$ is, after normalization, a determinantal point process induced by the projection operator onto the subspace $\sqrt{g}L$; the precise statement is recalled in Proposition 9.3 in the Appendix.

Informally, if g is such that the subspace $\sqrt{g}L$ no longer lies in L_2, then the measure $\Psi_g\mathbb{P}_\Pi$ ceases to be finite, and one obtains, precisely, an infinite determinantal measure corresponding to a subspace of locally square-integrable functions, one of the main constructions of this paper, see Theorem 2.11.

The Bessel point process of Tracy and Widom, which governs the ergodic decomposition of finite Pickrell measures, is the scaling limit of Jacobi orthogonal polynomial ensembles. In the problem of ergodic decomposition of infinite Pickrell measures one is led to investigating the scaling limit of infinite analogues of Jacobi orthogonal polynomial ensembles. The resulting scaling limit, an infinite determinantal measure, is computed in the paper and called the infinite Bessel point process; see Subsection 1.4 of this Introduction for the precise definition.

The main result of the paper, Theorem 1.11, identifies the ergodic decomposition measure of an infinite Pickrell measure with the infinite Bessel point process.

1.2. **Historical remarks.** Pickrell measures were introduced by Pickrell [33] in 1987. Borodin and Olshanski [6] studied in 2000 a closely related two-parameter family of measures on the space of infinite Hermitian matrices invariant with respect to the natural action of the infinite unitary group by conjugation; since the existence of such measures, as well as that of the original family considered by Pickrell, is proved by a computation that goes back to the work of Hua Loo-Keng [19], Borodin and Olshanski gave to the measures of their family the name of *Hua-Pickrell measures*. For various generalizations of Hua-Pickrell measures, see e.g. Neretin [26], Bourgade-Nikehbali-Rouault [8]. While Pickrell only considered values of the parameter for which the resulting measures are finite, Borodin and Olshanski [6] showed that the inifnite Pickrell and Hua-Pickrell measures are also well-defined. Borodin and Olshanski [6] proved that the ergodic decomposition of Hua-Pickrell probability measures is given by determinantal point processes that arise as scaling limits of pseudo-Jacobian orthogonal polynomial ensembles and posed the problem of describing the ergodic decomposition of infinite Hua-Pickrell measures.

The aim of this paper, devoted to Pickrell's original model, is to give an explicit description for the ergodic decomposition of infinite Pickrell measures on spaces of infinite complex matrices.

1.3. **Organization of the paper.** The paper is organized as follows. In the Introduction, we proceed by illustrating the main construction of the paper, that of infinite determinantal measures, on the specific example of the infinite Bessel point process. Next we recall the construction of Pickrell measures and the Olshanski-Vershik approach to Pickrell's classification of ergodic unitarily-invariant measures on the space of infinite complex matrices. We then formulate the main result of the paper, Theorem 1.11, which identifies the ergodic decomposition measure of an infinite Pickrell measure with the infinite Bessel point process (subject to the change of variable $y = 4/x$). We conclude the Introduction by giving an outline of the proof of Theorem 1.11: the ergodic decomposition measures of Pickrell measures are obtained as scaling limits of their finite-dimensional approximations, the radial parts of finite-dimensional projections of Pickrell measures. First, Lemma 1.14 states that the rescaled radial parts, multiplied by a certain positive density, converge to the desired ergodic decomposition measure multiplied by the same density. Second, it will develop that the normalized products of the push-forwards of rescaled radial parts to the space of configurations on the half-line with a suitably chosen multiplicative functional on the space of configurations, converge weakly in the space of measures on the space of configurations. Combining these statements will allow to conclude the proof of Theorem 1.11 in the last section of the paper.

Section 2 is devoted to the general construction of infinite determinantal measures on the space $\mathrm{Conf}(E)$ of configurations on a locally compact complete metric space E endowed with a sigma-finite Borel measure μ.

Start with a space H of functions on E locally square-integrable with respect to μ and an increasing collection of subsets

$$E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots$$

in E such that for any $n \in \mathbb{N}$ the restricted subspace $\chi_{E_n} H$ is a closed subspace in $L_2(E, \mu)$. If the corresponding projection operator Π_n is locally-trace class, then, by the Macchì-Soshnikov Theorem, the projection operator Π_n induces a determinantal measure $\mathbb{P}_n$ on $\mathrm{Conf}(E)$. Under certain additional assumptions it follows from the result of [11] (see Corollary 9.5 in the Appendix) that the measures $\mathbb{P}_n$ satisfy the following consistency property: if $\mathrm{Conf}(E, E_n)$ stands for the subset of those configurations all whose particles lie in E_n, then for any $n \in \mathbb{N}$ we have

$$\frac{\mathbb{P}_{n+1}|_{\mathrm{Conf}(E,E_n)}}{\mathbb{P}_{n+1}(\mathrm{Conf}(E, E_n))} = \mathbb{P}_n \tag{3}$$

The consistency property (3) implies that there exists a sigma-finite measure $\mathbb{B}$ such that for any $n \in \mathbb{N}$ we have $0 < \mathbb{B}(\mathrm{Conf}(E, E_n)) < +\infty$ and

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E,E_n)}}{\mathbb{B}(\mathrm{Conf}(E, E_n))} = \mathbb{P}_n$$

The measure $\mathbb{B}$ is called an infinite determinantal measure. An alternative description of infinite determinantal measures uses the formalism of multiplicative functionals. In [11] it is proved in (see also [12] and Proposition 9.3 in the Appendix) that a determinantal measure times an integrable multiplicative functional is, after normalization, again a determinantal measure. Now, if one takes the product of a determinantal measure by a convergent, but not integrable, multiplicative functional, then one obtains an infinite determinantal measure. This reduction of infinite determinantal measure to usual ones by taking the product with a multiplicative functional is essential for the proof of Theorem 1.11. Section 2 is concluded by the proof of the existence of the infinite Bessel point process.

Section 3 studies convergence of determinantal probability measures given by positive contractions that are locally trace-class. We start by recalling that locally trace-class convergence of operators implies weak convergence of the corresponding determinantal measures in the space of probability measures on the space of configurations. In the study of infinite Pickrell measures, we need to consider induced processes of the Bessel point process as well as as finite-rank perturbations of the Bessel point process, and in Section 3 sufficient conditions are given for the convergence of induced processes and of processes induced by finite-rank perturbations. We conclude Section 3 by establishing, for infinite determinantal measures obtained as finite-rank perturbations, the convergence of the family of determinantal processes obtained by inducing on an exhausting family of subsets of the phase space to the initial, unperturbed, determinantal process.

In Section 4, we embed suitable subsets of the space of configurations into the space of *finite* measures on the phase space E and give sufficient conditions for precompactness of families of determinantal measures with respect to the weak topology on the space of finite measures on the space of *finite* measures on E (which is stronger than the usual weak topology on the space of finite measures on the space of *Radon* measures, equivalent to the weak topology on the space of finite measures on the space of configurations). This step is needed for proving the vanishing of the "Gaussian parameter" for the ergodic components of Pickrell measures. Borodin and Olshanski [6] proved this vanishing for the ergodic components of Hua-Pickrell measures: in fact, the estimate of their argument can be interpreted as the assertion of *tightness* of the family of rescaled radial parts of Hua-Pickrell measures considered as measures in the space of finite measures on the space of *finite* measures. We next study weak convergence of induced processes and of finite-rank perturbations with respect to the new topology.

In Section 5, we go back to radial parts of Pickrell measures. We start by recalling the determinantal representation for radial parts of finite Pickrell measures and the convergence of the resulting determinantal processes to the modified Bessel point process (the usual Bessel point process of Tracy and Widom [43] subject to the change of variable $y = 4/x$). Next, we represent the radial parts of infinite Pickrell measures as infinite determinantal measures corresponding to finite-rank perturbations of Jacobi orthogonal polynomial ensembles. The main result of this section is Proposition 5.5 which shows that the scaling limit of the infinite determinantal measures corresponding to the radial parts of infinite Pickrell measures is precisely the modified infinite Bessel point process of the Introduction. Infinite determinantal measures are made finite by taking the product with a suitable multiplicative functional, and weak convergence is established both in the space of finite measures on the space of configurations and in the space of finite measures in the space of finite measures. The latter statement will be essential in the proof of the vanishing of the "Gaussian parameter" in the following Section.

In Section 6, we pass from the convergence, in the space of finite measures on the space of configurations and in the space of finite measures in the space of finite measures, of rescaled radial parts of Pickrell measures to the convergence, in the space of finite measures on the Pickrell set, of finite-dimensional approximations of Pickrell measures. In particular, in this section we establish the vanishing of the "Gaussian parameter" for ergodic components of infinite Pickrell measures. Proposition 6.1 proved in this section allows us to complete the proof of Proposition 1.16.

The final Section 7 is mainly devoted to the proof of Lemma 1.14, which relies on the well-known asymptotics of the Harish-Chandra-Itzykson-Zuber orbital integrals. Combining Lemma 1.14 with Proposition 1.16, we conclude the proof of Theorem 1.11.

The paper has three appendices. In Appendix A, we collect the needed facts about the Jacobi orthogonal polynomials, including the recurrence relation between the n-th Christoffel-Darboux kernel corresponding to parameters (α, β) and the $n-1$-th Christoffel-Darboux kernel corresponding to parameters $(\alpha+2, \beta)$. Appendix B is devoted to determinantal point processes on spaces of configurations. We start by recalling the definition of the space of configurations, its Borel structure and its topology; we next introduce dterminantal point processes, recall the Macchì-Soshnikov Theorem and the rule of transformation of kernels under a change of variables. We next recall the definition of multiplicative functionals on the space of configurations, formulate the result of [11] (see also [12]) that a determinantal point process times a multiplicative functional is again a determinantal point process and give an explicit representation of the resulting kernels; in particular, we recall the representation from [11], [12] for kernels of induced processes. Finally, in Appendix C we recall the construction of Pickrell measures following a computation of Hua Loo-Keng [19] as well as the observation of Borodin and Olshanski [6] in the infinite case and then, using Kakutani's Theorem in the same way as Borodin and Olshanski [6], prove that Pickrell measures corresponding to distrinct values of the parameter s are mutually singular.

1.4. **The Infinite Bessel Point Process.**

1.4.1. *Outline of the construction.* Take $n \in \mathbb{N}$, $s \in \mathbb{R}$, and endow the cube $(-1,1)^n$ with the measure

$$\prod_{1\le i<j\le n}(u_i-u_j)^2\prod_{i=1}^{n}(1-u_i)^s du_i. \tag{4}$$

For $s > -1$, the measure (4) is the Jacobi orthogonal polynomial ensemble, a determinantal point process induced by the n-th Christoffel-Darboux projection operator for Jacobi polynomials. The classical Heine-Mehler of Jacobi polynomials implies an asymptotics for the Christoffel-Dabroux kernels and, consequently, also for the corresponding determinantal point processes, whose scaling limit, with respect to the scaling

$$u_i = 1-\frac{y_i}{2n^2}, i=1,\dots,n, \tag{5}$$

is the Bessel point process of Tracy and Widom [43]. Recall that the Bessel point process is governed by the projection operator, in $L_2((0,+\infty),\mathrm{Leb})$, onto the subspace of functions whose Hankel transform is supported in $[0,1]$.

For $s \le -1$, the measure (4) is infinite. To describe its scaling limit, we start by recalling a recurrence relation between Christoffel-Darboux kernels of Jacobi polynomials and the consequent relation between the corresponding orthogonal polynomial ensembles: namely, the n-th Christoffel-Darboux kernel of the Jacobi ensemble with parameter s is a rank one perturbation of the $n-1$-th Christoffel-Darboux kernel of the Jacobi ensemble corresponding to parameter $s+2$.

This recurrence relation motivates the following construction. Consider the range of the Christoffel-Darboux projection operator. It is a finite-dimensional subspace of polynomials of degree less than n multiplied by the weight $(1-u)^{s/2}$. Consider the same subspace for $s \leq -1$. The resulting space is no longer a subspace of L_2; it is nonetheless a well-defined space of *locally* square-integrable functions. In view of the recurrence relation, furthermore, our subspace corresponding to the parameter s is a rank one perturbation of a similar subspace corresponding to parameter $s+2$, and so on, until we arrive at a value of the parameter, denoted $s+2n_s$ in what follows, for which the subspace becomes part of L_2. Our initial subspace is thus a finite-rank perturbation of a closed subspace in L_2 such that the rank of the perturbation depends on s but not on n. Now we take this representation to the scaling limit and obtain a subspace of locally square-integrable functions on $(0,+\infty)$, which, again, is a finite-rank perturbation of the range of the Bessel projection operator corresponding to the parameter $s+2n_s$.

To such a subspace of locally square-integable functions we next assign a sigma-finite measure on the space of configurations, the *infinite Bessel point process*. The infinite Bessel point process is the scaling limit of the measures (4) under the scaling (5).

1.4.2. *The Jacobi orthogonal polynomial ensemble.* First let $s > -1$. Let $P_n^{(s)}$ be the standard Jacobi orthogonal polynomials corresponding to the weight $(1-u)^s$. Let $\tilde{K}_n^{(s)}(u_1,u_2)$ the n-th Christoffel-Darboux kernel of the Jacobi orthogonal polynomial ensemble, see formulas (113), (114) in the Appendix. We now have the following well-known determinantal representation for the measure (4) in the case $s > -1$:

(6)
$$\mathrm{const}_{n,s} \prod_{1\leq i<j\leq n} (u_i-u_j)^2 \prod_{i=1}^{n} (1-u_i)^s du_i = \frac{1}{n!} \det \tilde{K}_n^{(s)}(u_i,u_j) \cdot \prod_{i=1}^{n} du_i,$$

where the normalization constant $\mathrm{const}_{n,s}$ is chosen in such a way that the left-hand side be a probability measure .

1.4.3. *The recurrence relation for Jacobi orthogonal polynomial ensembles.* We write Leb for the usual Lebesgue measure on the real line or on its subset. Given a finite family of functions $f_1,\dots,f_N$ on the real line, let $\mathrm{span}(f_1,\dots,f_N)$ stand for the vector space these functions span. The Christoffel-Darboux kernel $\tilde{K}_n^{(s)}$ is the kernel of the opertaor of orthogonal projection, in the space $L_2([-1,1],\mathrm{Leb})$, onto the subspace

$$(7)\quad L_{Jac}^{(s,n)} = \mathrm{span}\left((1-u)^{s/2}, (1-u)^{s/2}u, \dots, (1-u)^{s/2}u^{n-1}\right) = \\ = \mathrm{span}\left((1-u)^{s/2}, (1-u)^{s/2+1}, \dots, (1-u)^{s/2+n-1}\right).$$

By definition, we have a direct-sum decomposition

$$L_{Jac}^{(s,n)} = \mathbb{C}(1-u)^{s/2} \oplus L_{Jac}^{(s+2,n-1)}$$

By Proposition 8.1 in the Appendix, for any $s > -1$ we have the recurrence relation

$$(8)\quad \tilde{K}_n^{(s)}(u_1,u_2) = \frac{s+1}{2^{s+1}} P_{n-1}^{(s+1)}(u_1)(1-u_1)^{s/2} P_{n-1}^{(s+1)}(u_2)(1-u_2)^{s/2} + \\ + \tilde{K}_n^{(s+2)}(u_1,u_2)$$

and, consequently, an orthogonal direct-sum decomposition

$$L_{Jac}^{(s,n)} = \mathbb{C}P_{n-1}^{(s+1)}(u)(1-u)^{s/2} \oplus L_{Jac}^{(s+2,n-1)}.$$

We now pass to the case $s \le -1$. Define a natural number n_s by the relation

$$\frac{s}{2} + n_s \in \left(-\frac{1}{2}, \frac{1}{2}\right]$$

and introduce the subspace

(9)

$$\tilde{V}^{(s,n)} = \mathrm{span}\left((1-u)^{s/2}, (1-u)^{s/2+1}, \dots, P_{n-n_s}^{(s+2n_s-1)}(u)(1-u)^{s/2+n_s-1}\right).$$

By definition, we have is a direct sum decomposition

$$(10)\qquad L_{Jac}^{(s,n)} = \tilde{V}^{(s,n)} \oplus L_{Jac}^{(s+2n_s,n-n_s)}.$$

Note here that

$$L_{Jac}^{(s+2n_s,n-n_s)} \subset L_2([-1,1],\mathrm{Leb}),$$

while

$$\tilde{V}^{(s,n)} \bigcap L_2([-1,1],\mathrm{Leb}) = 0.$$

1.4.4. *Scaling limits.* Recall that the scaling limit, with respect to the scaling 5), of Christoffel-Darboux kernels $\tilde{K}_n^{(s)}$ of the Jacobi orthogonal polynomial ensemble, is given by the Bessel kernel $\tilde{J}_s$ of Tracy and Widom [43] (the definition of the Bessel kernel is recalled in the Appendix and the precise statement on the scaling limit is recalled in Proposition 8.3 in the Appendix).

It is clear that, for any β, under the scaling (5), we have

$$\lim_{n\to\infty} (2n^2)^\beta (1-u_i)^\beta = y_i^\beta$$

and, for any $\alpha > -1$, by the classical Heine-Mehler asymptotics for Jacobi polynomials, we have

$$\lim_{n\to\infty} 2^{-\frac{\alpha+1}{2}} n^{-1} P_n^{(\alpha)}(u_i)(1-u_i)^{\frac{\alpha-1}{2}} = \frac{J_\alpha(\sqrt{y_i})}{\sqrt{y_i}}.$$

It is therefore natural to take the subspace

$$\tilde{V}^{(s)} = \mathrm{span}\left(y^{s/2}, y^{s/2+1}, \dots, \frac{J_{s+2n_s-1}(\sqrt{y})}{\sqrt{y}} \right). \tag{11}$$

as the scaling limit of the subspace (9).

Furthermore, we already know that the scaling limit of the subspace (10) is the subspace $\tilde{L}^{(s+2n_s)}$, the range of the operator $\tilde{J}_{s+2n_s}$.

We arrive at the subspace $\tilde{H}^{(s)}$

$$\tilde{H}^{(s)} = \tilde{V}^{(s)} \oplus \tilde{L}^{(s+2n_s)}. \tag{12}$$

It is natural to consider the subspace $\tilde{H}^{(s)}$ as the scaling limit of the subspaces $L_{Jac}^{(s,n)}$ under the scaling (5) as $n \to \infty$.

Note that the subspace $\tilde{H}^{(s)}$ consists of locally square-integrable functions, which, moreover, only fail to be square-integrable *at zero*: for any $\varepsilon > 0$, the subspace $\chi_{[\varepsilon,+\infty)}\tilde{H}^{(s)}$ is contained in L_2.

1.4.5. *Definition of the infinite Bessel point process.* We now proceed to a precise description, in this specific case, of one of the main constructions of the paper: that of a sigma-finite measure $\tilde{\mathbb{B}}^{(s)}$, the scaling limit of infinite Jacobi ensembles (4) under the scaling (5). Let $\mathrm{Conf}((0,+\infty))$ be the space of configurations on $(0,+\infty)$. Given a Borel subset $E_0 \subset (0,+\infty)$, we let $\mathrm{Conf}((0,+\infty), E_0)$ be the subspace of configurations all whose particles lie in E_0. Generally, given a measure $\mathbb{B}$ on a set X and a measurable subset $Y \subset X$ such that $0 < \mathbb{B}(Y) < +\infty$, we let $\mathbb{B}|_Y$ stand for the restriction of the measure $\mathbb{B}$ onto the subset Y.

It will be proved in what follows that, for any $\varepsilon > 0$, the subspace $\chi_{(\varepsilon,+\infty)}\tilde{H}^{(s)}$ is a closed subspace of $L_2((0,+\infty),\mathrm{Leb})$ and that the operator $\tilde{\Pi}^{(\varepsilon,s)}$ of orthogonal projection onto the subspace $\chi_{(\varepsilon,+\infty)}\tilde{H}^{(s)}$ is locally of trace class. By the Macchì-Soshnikov Theorem, the operator $\tilde{\Pi}^{(\varepsilon,s)}$ induces a determinantal measure $\mathbb{P}_{\tilde{\Pi}^{(\varepsilon,s)}}$ on $\mathrm{Conf}((0,+\infty))$.

Proposition 1.1. *Let $s \le -1$. There exists a sigma-finite measure $\mathbb{B}^{(s)}$ on $\mathrm{Conf}((0,+\infty))$ such that we have*

1. *the particles of $\mathbb{B}$-almost every configuration do not accumulate at zero;*
2. *for any $\varepsilon > 0$ we have*

$$0 < \mathbb{B}(\mathrm{Conf}((0,+\infty);(\varepsilon,+\infty)) < +\infty$$

and

$$\frac{\mathbb{B}|_{\mathrm{Conf}((0,+\infty);(\varepsilon,+\infty))}}{\mathbb{B}(\mathrm{Conf}((0,+\infty);(\varepsilon,+\infty))}=\mathbb{P}_{\tilde{\Pi}^{(\varepsilon,s)}}.$$

These conditions define the measure $\tilde{\mathbb{B}}^{(s)}$ uniquely up to multiplication by a constant.

Remark. For $s\neq -1,-3,\ldots$, we can also write

$$\tilde{H}^{(s)}=\mathrm{span}(y^{s/2},\ldots,y^{s/2+n_s-1})\oplus\tilde{L}^{(s+2n_s)}$$

and use the preceding construction otherwise without change. For $s=-1$ note that the function $y^{1/2}$ fails to be square-integrable at infinity — whence the need for the definition given above. For $s>-1$, write $\tilde{\mathbb{B}}^{(s)}=\mathbb{P}_{\tilde{J}_s}$.

Proposition 1.2. *If $s_1\neq s_2$, then the measures $\tilde{\mathbb{B}}^{(s_1)}$ and $\tilde{\mathbb{B}}^{(s_2)}$ are mutully singular.*

The proof of Proposition 1.2 will be derived from Proposition 1.4, which in turn, will be obtained as a corollary of the main result, Theorem 1.11, in the last section of the paper.

1.5. **The modified Bessel point process.** In what follows, we will need the Bessel point process subject to the change of variable $y=4/x$. We thus consider the half-line $(0,+\infty)$ endowed with the standard Lebesgue measure Leb. Take $s>-1$ and introduce a kernel $J^{(s)}$ by the formula

(13)

$$J^{(s)}(x_1,x_2)=\frac{J_s\left(\frac{2}{\sqrt{x_1}}\right)\frac{1}{\sqrt{x_2}}J_{s+1}\left(\frac{2}{\sqrt{x_2}}\right)-J_s\left(\frac{2}{\sqrt{x_2}}\right)\frac{1}{\sqrt{x_1}}J_{s+1}\left(\frac{2}{\sqrt{x_1}}\right)}{x_1-x_2},$$
$$x_1>0,\ x_2>0\,.$$

or, equivalently,

$$J^{(s)}(x,y)=\frac{1}{x_1x_2}\int\limits_0^1 J_s\left(2\sqrt{\frac{t}{x_1}}\right)J_s\left(2\sqrt{\frac{t}{x_2}}\right)dt. \tag{14}$$

The change of variable $y=4/x$ reduces the kernel $J^{(s)}$ to the kernel $\tilde{J}_s$ of the Bessel point process of Tracy and Widom considered above (recall here that a change of variables $u_1=\rho(v_1)$, $u_2=\rho(v_2)$ transforms a kernel $K(u_1,u_2)$ to a kernel of the form $K(\rho(v_1),\rho(v_2))(\sqrt{\rho'(v_1)\rho'(v_2)})$). The kernel $J^{(s)}$ therefore induces on the space $L_2((0,+\infty),\mathrm{Leb})$ a locally trace class operator of orthogonal projection, for which, slightly abusing notation, we keep the symbol $J^{(s)}$; we denote $L^{(s)}$ the range of $J^{(s)}$. By the Macchì-Soshnikov Theorem, the operator $J^{(s)}$ induces a determinantal measure $\mathbb{P}_{J^{(s)}}$ on $\mathrm{Conf}((0,+\infty))$.

1.6. **The modified infinite Bessel point process.** The involutive homeomorphism

$$y = 4/x$$

of the half-line $(0, +\infty)$ induces a corresponding change of variable homeomorphism of the space $\mathrm{Conf}((0, +\infty))$. Let $\mathbb{B}^{(s)}$ be the image of $\widetilde{\mathbb{B}^{(s)}}$ under our change of variables. As we shall see below, the measure $\mathbb{B}^{(s)}$ is precisely the ergodic decomposition measure for the infinite Pickrell measures.

A more explicit description description of the measure $\mathbb{B}^{(s)}$ can be given as follows.

By definition, we have

$$L^{(s)} = \left\{ \frac{\varphi(4/x)}{x}, \varphi \in \tilde{L}^{(s)} \right\}.$$

(the behaviour of determinantal measures under a change of variables is recalled in the Appendix).

We similarly let $V^{(s)}, H^{(s)} \subset L_{2,\mathrm{loc}}((0, +\infty), \mathrm{Leb})$ be the images of the subspaces $\tilde{V}^{(s)}$, $\tilde{H}^{(s)}$ under our change of variables $y = 4/x$:

$$V^{(s)} = \left\{ \frac{\varphi(4/x)}{x}, \varphi \in \tilde{V}^{(s)} \right\}, \ H^{(s)} = \left\{ \frac{\varphi(4/x)}{x}, \varphi \in \tilde{H}^{(s)} \right\}.$$

By definition, we have

$$V^{(s)} = \mathrm{span}\left(x^{-s/2-1}, \dots, \frac{J_{s+2n_s-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}} \right). \tag{15}$$

$$H^{(s)} = V^{(s)} \oplus L^{(s+2n_s)}. \tag{16}$$

It will develop that for all $R > 0$ the subspace $\chi_{(0,R)} H^{(s)}$ is a closed subspace in $L_2((0, +\infty), \mathrm{Leb})$; let $\Pi^{(s,R)}$ be the corresponding orthogonal projection operator. By definition, the operator $\Pi^{(s,R)}$ is locally of trace-class and, by the Macchì-Soshnikov Theorem, the operator $\Pi^{(s,R)}$ induces a determinantal measure $\mathbb{P}_{\Pi^{(s,R)}}$ on $\mathrm{Conf}((0, +\infty))$.

The measure $\mathbb{B}^{(s)}$ is characterized by the following conditions:

(1) the set of particles of $\mathbb{B}^{(s)}$-almost every configuration is bounded;
(2) for all $R > 0$ we have

$$0 < \mathbb{B}(\mathrm{Conf}((0, +\infty); (0, R)) < +\infty$$

and

$$\frac{\mathbb{B}|_{\mathrm{Conf}((0,+\infty);(0,R))}}{\mathbb{B}(\mathrm{Conf}((0, +\infty); (0, R))} = \mathbb{P}_{\Pi^{(s,R)}}.$$

These conditions define the measure $\mathbb{B}^{(s)}$ uniquely up to multiplication by a constant.

Remark. For $s \neq -1, -3, \dots$, we can of course also write

$$H^{(s)} = \mathrm{span}(x^{-s/2-1}, \dots, x^{-s/2-n_s+1}) \oplus L^{(s+2n_s)}.$$

Let $\mathscr{I}_{1,\mathrm{loc}}((0,+\infty),\mathrm{Leb})$ be the space of locally trace-class operators acting on the space $L_2((0,+\infty),\mathrm{Leb})$ (see the Appendix for the detailed definition). We have the following proposition describing the asymptotic behaviour of the operators $\Pi^{(s,R)}$ as $R \to \infty$.

Proposition 1.3. *Let* $s \leq -1$. *Then*

(1) *as* $R \to \infty$ *we have*

$$\Pi^{(s,R)} \to J^{(s+2n_s)}$$

in $\mathscr{I}_{1,\mathrm{loc}}((0,+\infty),\mathrm{Leb})$*;*

(2) *Consequently, as* $R \to \infty$*, we have*

$$\mathbb{P}_{\Pi^{(s,R)}} \to \mathbb{P}_{J^{(s+2n_s)}}$$

weakly in the space of probability measures on $\mathrm{Conf}((0,+\infty))$.

As before, for $s > -1$, write $\mathbb{B}^{(s)} = \mathbb{P}_{J^{(s)}}$. Proposition 1.2 is equivalent to the following

Proposition 1.4. *If* $s_1 \neq s_2$*, then the measures* $\mathbb{B}^{(s_1)}$ *and* $\mathbb{B}^{(s_2)}$ *are mutully singular.*

Proposition 1.4 will be obtained as the corollary of the main result, Theorem 1.11, in the last section of the paper.

We now represent the measure $\mathbb{B}^{(s)}$ as the product of a determinantal probability measure and a multiplicative functional. Here we limit ourselves to specific example of such a representation, but in what follows we will see that they can be constructed in much greater generality. Introduce a function S on the space $\mathrm{Conf}((0,+\infty))$ by setting

$$S(X) = \sum_{x \in X} x.$$

The function S may, of course, assume value ∞, but the set of such configurations is $\mathbb{B}^{(s)}$-negligible, as is shown by the following

Proposition 1.5. *For any* $s \in \mathbb{R}$ *we have* $S(X) < +\infty$ *almost surely with respect to the measure* $\mathbb{B}^{(s)}$ *and for any* $\beta > 0$ *we have*

$$\exp(-\beta S(X)) \in L_1(\mathrm{Conf}((0,+\infty)), \mathbb{B}^{(s)}).$$

Furthermore, we shall now see that the measure

$$\frac{\exp(-\beta S(X))\mathbb{B}^{(s)}}{\int\limits_{\mathrm{Conf}((0,+\infty))} \exp(-\beta S(X))d\mathbb{B}^{(s)}}$$

is determinantal.

Proposition 1.6. *For any $s \in \mathbb{R}$, $\beta > 0$, the subspace*

$$\exp\left(-\beta x/2\right) H^{(s)} \tag{17}$$

is a closed subspace of $L_2\big((0,+\infty),\mathrm{Leb}\big)$, and the operator of orthogonal projection onto the subspace (17) is locally of trace class.

Let $\Pi^{(s,\beta)}$ be the operator of orthogonal projection onto the subspace (17).

By Proposition 1.6 and the Macchì-Soshnikov Theorem, the operator $\Pi^{(s,\beta)}$ induces a determinantal probability measure on the space $\mathrm{Conf}((0,+\infty))$.

Proposition 1.7. *For any $s \in \mathbb{R}$, $\beta > 0$, we have*

$$\frac{\exp(-\beta S(X))\mathbb{B}^{(s)}}{\int\limits_{\mathrm{Conf}((0,+\infty))} \exp(-\beta S(X))d\mathbb{B}^{(s)}} = \mathbb{P}_{\Pi^{(s,\beta)}}. \tag{18}$$

1.7. **Unitarily-Invariant Measures on Spaces of Infinite Matrices.**

1.7.1. *Pickrell Measures.* Let $\mathrm{Mat}(n,\mathbb{C})$ be the space of $n \times n$ matrices with complex entries:

$$\mathrm{Mat}(n,\mathbb{C}) = \{z = (z_{ij}),\ i = 1,\dots,n; j = 1,\dots,n\}$$

Let $\mathrm{Leb} = dz$ be the Lebesgue measure on $\mathrm{Mat}(n,\mathbb{C})$. For $n_1 < n$, let

$$\pi^n_{n_1} : \mathrm{Mat}(n,\mathbb{C}) \to \mathrm{Mat}(n_1,\mathbb{C})$$

be the natural projection map that to a matrix $z = (z_{ij}), i,j = 1,\dots,n,$ assigns its upper left corner, the matrix $\pi^n_{n_1}(z) = (z_{ij}), i,j = 1,\dots,n_1$.

Following Pickrell [31], take $s \in \mathbb{R}$ and introduce a measure $\widetilde{\mu}^{(s)}_n$ on $\mathrm{Mat}(n,\mathbb{C})$ by the formula

$$\widetilde{\mu}^{(s)}_n = \det(1+z^*z)^{-2n-s}dz.$$

The measure $\widetilde{\mu}^{(s)}_n$ is finite if and only if $s > -1$.

The measures $\widetilde{\mu}^{(s)}_n$ have the following property of consistency with respect to the projections $\pi^n_{n_1}$.

Proposition 1.8. *Let $s \in \mathbb{R}$, $n \in \mathbb{N}$ satisfy $n+s>0$. Then for any $\tilde z \in \mathrm{Mat}(n,\mathbb{C})$ we have*

$$\int\limits_{(\pi_n^{n+1})^{-1}(\tilde z)} \det(1+z^*z)^{-2n-2-s}dz = \tag{19}$$

$$= \frac{\pi^{2n+1}(\Gamma(n+1+s))^2}{\Gamma(2n+2+s)\cdot\Gamma(2n+1+s)} \det(1+\tilde z^*\tilde z)^{-2n-s}.$$

Now let $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ be the space of infinite matrices whose rows and columns are indexed by natural numbers and whose entries are complex:

$$\mathrm{Mat}(\mathbb{N},\mathbb{C}) = \{z=(z_{ij}), i,j\in\mathbb{N}, z_{ij}\in\mathbb{C}\}.$$

Let $\pi_n^\infty : \mathrm{Mat}(\mathbb{N},\mathbb{C}) \to \mathrm{Mat}(n,\mathbb{C})$ be the natural projection map that to an infinite matrix $z\in\mathrm{Mat}(\mathbb{N},\mathbb{C})$ assigns its upper left $n\times n$-“corner”, the matrix $(z_{ij}), i,j=1,\dots,n$.

For $s>-1$, Proposition 1.8 together with the Kolmogorov Existence Theorem [20] implies that there exists a unique probability measure $\mu^{(s)}$ on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that for any $n\in\mathbb{N}$ we have the relation

$$(\pi_n^\infty)_*\mu^{(s)} = \pi^{-n^2}\prod_{l=1}^{n}\frac{\Gamma(2l+s)\Gamma(2l-1+s)}{(\Gamma(l+s))^2}\widetilde{\mu}_n^{(s)}.$$

If $s\le -1$, then Proposition 1.8 together with the Kolmogorov Existence Theorem [20] implies that for any $\lambda>0$ there exists a unique infinite measure $\mu^{(s,\lambda)}$ on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that

(1) for any $n\in\mathbb{N}$ satisfying $n+s>0$ and any compact subset $Y\subset\mathrm{Mat}(n,\mathbb{C})$ we have $\mu^{(s,\lambda)}(Y)<+\infty$; the pushforwards $(\pi_n^\infty)_*\mu^{(s,\lambda)}$ are consequently well-defined;
(2) for any $n\in\mathbb{N}$ satisfying $n+s>0$ we have

$$(\pi_n^\infty)_*\mu^{(s,\lambda)} = \lambda\left(\prod_{l=n_0}^{n}\pi^{-2n}\frac{\Gamma(2l+s)\Gamma(2l-1+s)}{(\Gamma(l+s))^2}\right)\widetilde{\mu}^{(s)}. \tag{20}$$

The measures $\mu^{(s,\lambda)}$ will be called *infinite Pickrell measures*. Slightly abusing notation, we shall omit the super-script λ and write $\mu^{(s)}$ for a measure defined up to a multiplicative constant. See p.116 in Borodin and Olshanski [6] for a detailed presentation of infinite Pickrell measures.

Proposition 1.9. *For any $s_1,s_2\in\mathbb{R}$, $s_1\ne s_2$, the Pickrell measures $\mu^{(s_1)}$ and $\mu^{(s_2)}$ are mutually singular.*

Proposition 1.9 is obtained from Kakutani's Theorem in the spirit of [6], see also [26].

Let $U(\infty)$ be the infinite unitary group: an infinite matrix $u = (u_{ij})_{i,j\in\mathbb{N}}$ belongs to $U(\infty)$ if there exists a natural number n_0 such that the matrix

$$(u_{ij}), i, j \in [1, n_0]$$

is unitary, while $u_{ii} = 1$ if $i > n_0$ and $u_{ij} = 0$ if $i \neq j$, $\max(i, j) > n_0$.

The group $U(\infty) \times U(\infty)$ acts on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ by multiplication on both sides:

$$T_{(u_1,u_2)}z \;=\; u_1 z u_2^{-1}.$$

The Pickrell measures $\mu^{(s)}$ are by definition $U(\infty) \times U(\infty)$-invariant. For the rôle of Pickrell and related mesures in the representation theory of $U(\infty)$, see [28], [29], [30].

Theorem 1 and Corollary 1 in [9] imply that the measures $\mu^{(s)}$ admit an ergodic decomposition, while Theorem 1 in [10] implies that for any $s \in \mathbb{R}$ the ergodic components of the measure $\mu^{(s)}$ are almost surely finite. We now formulate this result in greater detail. Recall that a $U(\infty) \times U(\infty)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ is called *ergodic* if every $U(\infty) \times U(\infty)$invariant Borel subset of $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ either has measure zero or has complement of measure zero. Equivalently, ergodic probability measures are extremal points of the convex set of all $U(\infty) \times U(\infty)$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. Let $\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ stand for the set of all ergodic $U(\infty) \times U(\infty)$-invariant probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. The set $\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ is a Borel subset of the set of all probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ (see, e.g., [9]). Theorem 1 in [10] implies that for any $s \in \mathbb{R}$ there exists a unique sigma-finite Borel measure $\overline{\mu}^{(s)}$ on the set $\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}))$ such that we have

$$\mu^{(s)} = \int\limits_{\mathfrak{M}_{erg}(\mathrm{Mat}(\mathbb{N},\mathbb{C})} \eta d\overline{\mu}^{(s)}(\eta). \tag{21}$$

The main result of this paper is an explicit description of the measure $\overline{\mu}^{(s)}$ and its identification, after a change of variable, with the infinite Bessel point process considered above.

1.8. **Classification of ergodic measures.** First, we recall the classification of ergodic probability $U(\infty) \times U(\infty)$-invariant measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$. This classification has been obtained by Pickrell [31], [32]; Vershik [44] and Olshanski and Vershik [30] proposed a different approach to this classification in the case of unitarily-invariant measures on the space of infinite Hermitian matrices, and Rabaoui [34], [35] adapted the Olshanski-Vershik approach to the initial problem of Pickrell. In this note, the Olshanski-Vershik approach is followed as well.

Take $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$, denote $z^{(n)} = \pi_n^\infty z$, and let

$$\lambda_1^{(n)} \geqslant \ldots \geqslant \lambda_n^{(n)} \geqslant 0 \tag{22}$$

be the eigenvalues of the matrix

$$\left(z^{(n)}\right)^* z^{(n)},$$

counted with multiplicities, arranged in non-increasing order. To stress dependence on z, we write $\lambda_i^{(n)} = \lambda_i^{(n)}(z)$.

Theorem. (1) *Let η be an ergodic Borel $U(\infty)\times U(\infty)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$. Then there exist non-negative real numbers*

$$\gamma \geqslant 0,\ \ x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant \ldots \geqslant 0\,,$$

satisfying $\gamma \geqslant \sum_{i=1}^{\infty} x_i$, such that for η-almost every $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$ and any $i \in \mathbb{N}$ we have:

$$x_i = \lim_{n\to\infty} \frac{\lambda_i^{(n)}(z)}{n^2},\ \ \gamma = \lim_{n\to\infty} \frac{\mathrm{tr}\left(z^{(n)}\right)^* z^{(n)}}{n^2}. \tag{23}$$

(2) *Conversely, given non-negative real numbers $\gamma \geqslant 0,\ \ x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant \ldots \geqslant 0$ such that*

$$\gamma \geqslant \sum_{i=1}^{\infty} x_i\,,$$

there exists a unique $U(\infty)\times U(\infty)$-invariant ergodic Borel probability measure η on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that the relations (23) hold for η-almost all $z \in \mathrm{Mat}(\mathbb{N},\mathbb{C})$.

Introduce the *Pickrell set* $\Omega_P \subset \mathbb{R}_+ \times \mathbb{R}_+^{\mathbb{N}}$ by the formula

$$\Omega_P = \left\{\omega = (\gamma, x)\colon x = (x_n),\ n \in \mathbb{N},\ x_n \geqslant x_{n+1} \geqslant 0,\ \gamma \geqslant \sum_{i=1}^{\infty} x_i\right\}.$$

The set Ω_P is, by definition, a closed subset of $\mathbb{R}_+ \times \mathbb{R}_+^{\mathbb{N}}$ endowed with the Tychonoff topology. For $\omega \in \Omega_P$ we let η_ω be the corresponding ergodic probability measure.

The Fourier transform of the measure η_ω is explicitly described as follows. First, for any $\lambda \in \mathbb{R}$ we have

$$\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \exp(i\lambda \Re z_{11}) d\eta_\omega(z) = \frac{\exp(-4(\gamma - \sum\limits_{k=1}^{\infty} x_k)\lambda^2)}{\prod\limits_{k=1}^{\infty}(1+4x_k\lambda^2)}. \tag{24}$$

Denote $F_\omega(\lambda)$ the expression in the right-hand side of (24); then, for any $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ we have

(25)
$$\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})} \exp(i(\lambda_1 \Re z_{11} + \dots + \lambda_m \Re z_{mm}))d\eta_\omega(z) = F_\omega(\lambda_1) \cdot \dots \cdot F_\omega(\lambda_m).$$

The Fourier transform is fully defined, and the measure η_ω is completely described. An explicit construction of the ergodic measures η_ω is given as follows. First, if one takes all entries of the matrix z are independent identically distributed complex Gaussian random variables with expectation 0 and variance $\tilde{\gamma}$, then the resulting Gaussian measure with parameter $\tilde{\gamma}$, clearly unitarily invariant and, by the Kolmogorov zero-one law, ergodic, corresponds to the parameter $\omega = (\tilde{\gamma}, 0, \dots, 0, \dots)$ — all x-coordinates are equal to 0 (indeed, singular values of a Gaussian matrix grow at rate $\sqrt{n}$ rather than n).

Next, let $(v_1, \dots, v_n, \dots)$, $(w_1, \dots, w_n, \dots)$ be two infinite independent vectors of independent identically distributed complex Gaussian random variables with variance $\sqrt{x}$, and set $z_{ij} = v_i w_j$. One thus obtains a measure whose unitary invariance is clear and whose ergodicity is immediate from the Kolmogorov zero-one law. This measure corresponds to the parameter $\omega \in \Omega_P$ such that $\gamma(\omega) = x$, $x_1(\omega) = x$, and all the other parameters are zero. Following Olshanski and Vershik [30], such measures are called *Wishart measures* with parameter x. In the general case, set $\tilde{\gamma} = \gamma - \sum\limits_{k=1}^{\infty} x_k$. The measure η_ω is then an infinite convolution of the Wishart measures with parameters $x_1, \dots, x_n, \dots$ and the Gaussian measure with parameter $\tilde{\gamma}$. Convergence of the series $x_1 + \dots + x_n + \dots$ ensures that the convolution is well-defined.

The quantity $\tilde{\gamma} = \gamma - \sum\limits_{k=1}^{\infty} x_k$ will therefore be called the *Gaussian parameter* of the measure η_ω. It will develop that the Gaussian parameter vanishes for almost all ergodic components of Pickrell measures.

By Proposition 3 in [9], the subset of ergodic $U(\infty) \times U(\infty)$-invariant measures is a Borel subset of the space of all Borel probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ endowed with the natural Borel structure (see, e.g., [3]). Furthermore, if one denotes η_ω the Borel ergodic probability measure corresponding to a point $\omega \in \Omega_P$, $\omega = (\gamma, x)$, then the correspondence

$$\omega \longrightarrow \eta_\omega$$

is a Borel isomorphism of the Pickrell set Ω_P and the set of $U(\infty) \times U(\infty)$-invariant ergodic probability measures on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$.

The Ergodic Decomposition Theorem (Theorem 1 and Corollary 1 of [9]) implies that each Pickrell measure $\mu^{(s)}$, $s \in \mathbb{R}$, induces a unique decomposing measure $\overline{\mu}^{(s)}$ on Ω_P such that we have

$$\mu^{(s)} = \int\limits_{\Omega_P} \eta_\omega \, d\overline{\mu}^{(s)}(\omega) \ . \tag{26}$$

The integral is understood in the usual weak sense, see [9].

For $s > -1$, the measure $\overline{\mu}^{(s)}$ is a probability measure on Ω_P, while for $s \leqslant -1$ the measure $\overline{\mu}^{(s)}$ is infinite.

Set

$$\Omega_P^0 = \{(\gamma, \{x_n\}) \in \Omega_P : x_n > 0 \quad \text{for all } n, \ \gamma = \sum_{n=1}^{\infty} x_n\}.$$

The subset Ω_P^0 is of course not closed in Ω_P.

Introduce a map

$$\mathrm{conf}\colon \Omega_P \to \mathrm{Conf}((0,+\infty))$$

that to a point $\omega \in \Omega_P, \omega = (\gamma, \{x_n\})$ assigns the configuration

$$\mathrm{conf}(\omega) = (x_1, \ldots, x_n, \ldots) \in \mathrm{Conf}((0,+\infty)).$$

The map $\omega \to \mathrm{conf}(\omega)$ is bijective in restriction to the subset Ω_P^0.

Remark. In the definition of the map conf, the “asymptotic eigenvalues” x_n are counted with multiplicities, while, if $x_{n_0} = 0$ for some n_0, then x_{n_0} and all subsequent terms are discarded, and the resulting configuration is finite. We shall see, however, that, $\overline{\mu}^{(s)}$-almost surely, all configurations are infinite and that, $\overline{\mu}^{(s)}$-almost surely, all multiplicities are equal to one. It will also develop that the complement $\Omega_P \setminus \Omega_P^0$ is $\overline{\mu}^{(s)}$-negligible for all s.

1.9. **Formulation of the main result.** We start by formulating the analogue of the Borodin-Olshanski Ergodic Decomposition Theorem [6] for finite Pickrell measures.

Proposition 1.10. *Let $s > -1$. Then $\overline{\mu}^{(s)}(\Omega_P^0) = 1$ and the $\overline{\mu}^{(s)}$-almost sure bijection $\omega \to \mathrm{conf}(\omega)$ identifies the measure $\overline{\mu}^{(s)}$ with the determinantal measure $\mathbb{P}_{J^{(s)}}$.*

The main result of this paper, an explicit description for the ergodic decomposition of infinite Pickrell measures, is given by the following

Theorem 1.11. *Let $s \in \mathbb{R}$, and let $\overline{\mu}^{(s)}$ be the decomposing measure, defined by (26), of the Pickrell measure $\mu^{(s)}$. Then*

(1) $\overline{\mu}^{(s)}(\Omega_P \setminus \Omega_P^0) = 0$;
(2) *the $\overline{\mu}^{(s)}$-almost sure bijection $\omega \to \mathrm{conf}(\omega)$ identifies $\overline{\mu}^{(s)}$ with the infinite determinantal measure $\mathbb{B}^{(s)}$.*

1.10. **A skew-product representation of the measure $\mathbb{B}^{(s)}$.** With respect to the measure $\mathbb{B}^{(s)}$, almost every configuration X only accumulates at zero and therefore admits a maximal particle that we denote $x_{\max}(X)$. We are interested in the distribution of the maximal particle under the measure $\mathbb{B}^{(s)}$. By definition, for any $R > 0$, the measure $\mathbb{B}^{(s)}$ assings finite weight to the set $\{X : x_{\max}(X) < R\}$. Furthermore, again by definition, for any $R > 0$ and $R_1, R_2 \le R$ we have the following relation:

$$\frac{\mathbb{B}^{(s)}\left(\{X : x_{\max}(X) < R_1\}\right)}{\mathbb{B}^{(s)}\left(\{X : x_{\max}(X) < R_2\}\right)} = \frac{\det\left(1 - \chi_{(R_1,+\infty)}\Pi^{(s,R)}\chi_{(R_1,+\infty)}\right)}{\det\left(1 - \chi_{(R_2,+\infty)}\Pi^{(s,R)}\chi_{(R_2,+\infty)}\right)}. \tag{27}$$

The push-forward of the measure $\mathbb{B}^{(s)}$ is a well-defined Borel sigma-finite measure on $(0, +\infty)$ for which we will use the symbol $\xi_{\max}\mathbb{B}^{(s)}$; the measure $\xi_{\max}\mathbb{B}^{(s)}$ is, of course, defined up to multiplication by a positive constant.

Question. What is the asymptotics of the quantity $\xi_{\max}\mathbb{B}^{(s)}(0, R)$ as $R \to \infty$? as $R \to 0$?

The operator $\Pi^{(s,R)}$ admits a kernel for which we keep the same symbol; consider the function $\varphi_R(x) = \Pi^{(s,R)}(x, R)$. By definition,

$$\varphi_R(x) \in \chi_{(0,R)}H^{(s)}.$$

Let $\overline{H}^{(s,R)}$ stand for the orthogonal complement to the one-dimensional subspace spanned by $\varphi_R(x)$ in $\chi_{(0,R)}H^{(s)}$. In other words, $\overline{H}^{(s,R)}$ is the subspace of those functions in $\chi_{(0,R)}H^{(s)}$ that assume value zero at the point R. Let $\overline{\Pi}^{(s,R)}$be the operator of orthogonal projection onto the subspace $\overline{H}^{(s,R)}$.

Proposition 1.12. *We have*

$$\mathbb{B}^{(s)} = \int_0^\infty \mathbb{P}_{\overline{\Pi}^{(s,R)}} d\xi_{\max}\mathbb{B}^{(s)}(R).$$

Proof. This immediately follows from the definition of the measure $\mathbb{B}^{(s)}$ and the characterization of Palm measures for determinantal point processes due to Shirai and Takahashi [38].

1.11. **The general scheme of ergodic decomposition.**

1.11.1. *Approximation.* Let $\mathfrak{F}$ be the family of σ-infinite $U(\infty) \times U(\infty)$-invariant measures μ on $\mathrm{Mat}(\mathbb{N}, \mathbb{C})$ for which there exists n_0 (dependent on μ) such that for all $R > 0$ we have

$$\mu(\left\{z \colon \max_{1 \leqslant i,j \leqslant n_0} |z_{ij}| < R\right\}) < +\infty.$$

By definition, all Pickrell measures belong to the class $\mathfrak{F}$.

We recall the result of [10] stating that every ergodic measure belonging to the class $\mathfrak{F}$ must be finite and that the ergodic components of any measure in $\mathfrak{F}$ are therefore almost surely finite (the existence of the ergodic decomposition for any measure $\mu \in \mathfrak{F}$ follows from the ergodic decomposition theorem for actions of inductively compact groups established in [9]). The classification of finite ergodic measures now implies that for every measure $\mu \in \mathfrak{F}$ there exists a unique Borel σ-finite measure $\overline{\mu}$ on the Pickrell set Ω_P such that

$$\mu = \int_{\Omega_P} \eta_\omega \, d\overline{\mu}(\omega). \tag{28}$$

Our next aim is to construct, following Borodin and Olshanski [6], a sequence of finite-dimensional approximations for the measure $\overline{\mu}$.

To a matrix $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ and a number $n \in \mathbb{N}$ assign the array

$$(\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_n^{(n)})$$

of eigenvalues arranged in non-increasing order of the matrix $(z^{(n)})^* z^{(n)}$, where

$$z^{(n)} = (z_{ij})_{i,j=1,\dots,n}.$$

For $n \in \mathbb{N}$ define a map

$$\mathfrak{r}^{(n)} \colon \mathrm{Mat}(\mathbb{N}, \mathbb{C}) \to \Omega_P$$

by the formula

$$\mathfrak{r}^{(n)}(z) = \left(\frac{1}{n^2} \mathrm{tr}(z^{(n)})^* z^{(n)}, \frac{\lambda_1^{(n)}}{n^2}, \frac{\lambda_2^{(n)}}{n^2}, \dots, \frac{\lambda_n^{(n)}}{n^2}, 0, 0, \dots \right). \tag{29}$$

It is clear by definition that for any $n \in \mathbb{N}$, $z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ we have

$$\mathfrak{r}^{(n)}(z) \in \Omega_P^0.$$

For any $\mu \in \mathfrak{F}$ and all sufficiently large $n \in \mathbb{N}$ the push-forwards $(\mathfrak{r}^{(n)})_* \mu$ are well-defined since the unitary group is compact. We shall presently see that for any $\mu \in \mathfrak{F}$ the measures $(\mathfrak{r}^{(n)})_* \mu$ approximate the ergodic decomposition measure $\overline{\mu}$.

We start by a direct description of the map that takes a measure $\mu \in \mathfrak{F}$ to its ergodic decomposition measure $\overline{\mu}$.

Following Borodin-Olshanski [6], let $\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N}, \mathbb{C})$ be the set of all matrices z such that

(1) for any k, there exists the limit $\lim\limits_{n\to\infty} \dfrac{1}{n^2} \lambda_n^{(k)} =: x_k(z)$;

(2) there exists the limit $\lim\limits_{n\to\infty} \dfrac{1}{n^2} \mathrm{tr}(z^{(n)})^* z^{(n)} =: \gamma(z)$.

Since the set of regular matrices has full measure with respect to any finite ergodic $U(\infty)\times U(\infty)$-invariant measure, the existence of the ergodic decomposition (28) implies

$$\mu(\mathrm{Mat}(\mathbb{N},\mathbb{C})\setminus\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C}))=0.$$

We introduce the map

$$\mathfrak{r}^{(\infty)}\colon \mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C})\to\Omega_P$$

by the formula

$$\mathfrak{r}^{(\infty)}(z)=(\gamma(z),x_1(z),x_2(z),\dots,x_k(z),\dots)\,.$$

The Ergodic Decomposition Theorem [9] and the classification of ergodic unitarily-invariant measures in the form of Olshanski and Vershik imply the important equality

$$(\mathfrak{r}^{(\infty)})_*\mu=\overline{\mu}. \tag{30}$$

Remark. This equality has a simple analogue in the context of De Finetti's theorem: in order to obtain the ergodic decomposition of an exchangeable measure on the space of binary sequences, one just needs to consider the push-forward of the initial measure by the almost-surely defined map that to each sequence assigns the frequency of zeros in it.

Given a complete separable metric space Z, we write $\mathfrak{M}_{\mathrm{fin}}(Z)$ for the space of all finite Borel measures on Z endowed with the weak topology. Recall [3] that $\mathfrak{M}_{\mathrm{fin}}(Z)$ is itself a complete separable metric space: the weak topology is induced, for instance, by the Lévy-Prohorov metric.

We proceed to showing that the measures $(\mathfrak{r}^{(n)})_*\mu$ approximate the measure $(\mathfrak{r}^{(\infty)})_*\mu=\overline{\mu}$ as $n\to\infty$. For finite measures μ the following statement is due to Borodin and Olshanski [6].

Proposition 1.13. *Let μ be a finite σ-invariant measure on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$. Then, as $n\to\infty$, we have*

$$(\mathfrak{r}^{(n)})_*\mu\to(\mathfrak{r}^{(\infty)})_*\mu$$

weakly in $\mathfrak{M}_{\mathrm{fin}}(\Omega_P)$.

Proof. Let $f\colon\Omega_P\to\mathbb{R}$ be continuous and bounded. For any $z\in\mathrm{Mat}_{\mathrm{reg}}(\mathbb{N},\mathbb{C})$, by definition, we have $\mathfrak{r}^{(n)}(z)\to\mathfrak{r}^{(\infty)}(z)$ as $n\to\infty$, and, consequently, also,

$$\lim_{n\to\infty}f(\mathfrak{r}^{(n)}(z))=f(\mathfrak{r}^{(\infty)}(z)),$$

whence, by bounded convergence theorem, we have

$$\lim_{n\to\infty}\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})}f(\mathfrak{r}^{(n)}(z))\,d\mu(z)=\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})}f(\mathfrak{r}^{(\infty)}(z))\,d\mu(z).$$

Changing variables, we arrive at the convergence

$$\lim_{n\to\infty}\int\limits_{\Omega_P} f(\omega)\, d(\mathfrak{r}^{(n)})_*\mu = \int\limits_{\Omega_P} f(\omega)\, d(\mathfrak{r}^{(\infty)})_*\mu,$$

and the desired weak convergence is established. □

For σ-finite measures $\mu \in \mathfrak{F}$, the Borodin-Olshanski proposition is modified as follows.

Lemma 1.14. *Let $\mu \in \mathfrak{F}$. There exists a positive bounded continuous function f on the Pickrell set Ω_P such that*

(1) $f \in L_1(\Omega_P, (\mathfrak{r}^{(\infty)})_*\mu)$ *and* $f \in L_1(\Omega_P, (\mathfrak{r}^{(n)})_*\mu)$ *for all sufficiently large* $n \in \mathbb{N}$*;*
(2) *as* $n \to \infty$*, we have*

$$f(\mathfrak{r}^{(n)})_*\mu \to f(\mathfrak{r}^{(\infty)})_*\mu$$

weakly in $\mathfrak{M}_{\text{fin}}(\Omega_P)$*.*

Proof of Lemma 1.14 will be given in Section 7.

Remark. As the above argument shows, the explicit characterization of the ergodic decomposition of Pickrell measures given in Theorem 1.11 does rely on the abstract result, Theorem 1 in [9], that a priori guarantees the existence of the ergodic decomposition and does not by itself give an alternative proof of the existence of the ergodic decomposition.

1.11.2. *Convergence of probability measures on the Pickrell set.* Recall that we have a natural forgetting map $\mathrm{conf} : \Omega_P \to \mathrm{Conf}(0, +\infty)$ that to a point $\omega = (\gamma, x)$, $x = (x_1, \dots, x_n, \dots)$, assigns the configuration $\mathrm{conf}(\omega) = (x_1, \dots, x_n, \dots)$.

For $\omega \in \Omega_P$, $\omega = (\gamma, x)$, $x = (x_1, \dots, x_n, \dots)$, $x_n = x_n(\omega)$, set

$$S(\omega) = \sum_{n=1}^{\infty} x_n(\omega).$$

In other words, we set $S(\omega) = S(\mathrm{conf}(\omega))$, and, slightly abusing notation, keep the same symbol for the new map. Take $\beta > 0$ and consider the measures

$$\exp(-\beta S(\omega))\mathfrak{r}^{(n)}(\mu^{(s)})),$$

$n \in \mathbb{N}$.

Proposition 1.15. *For any $s \in \mathbb{R}$, $\beta > 0$, we have*

$$\exp(-\beta S(\omega)) \in L_1(\Omega_P, \mathfrak{r}^{(n)}(\mu^{(s)})).$$

Introduce the probability measure

$$\nu^{(s,n,\beta)} = \frac{\exp(-\beta S(\omega))\mathfrak{r}^{(n)}(\mu^{(s)})}{\int\limits_{\Omega_P} \exp(-\beta S(\omega))d\mathfrak{r}^{(n)}(\mu^{(s)})}.$$

Now go back to the determinantal measure $\mathbb{P}_{\Pi^{(s,\beta)}}$ on the space $\mathrm{Conf}((0,+\infty))$ (cf. (18)) and let the measure $\nu^{(s,\beta)}$ on Ω_P be defined by the requirements

(1) $\nu^{(s,\beta)}(\Omega_P \setminus \Omega_P^0) = 0$;
(2) $\mathrm{conf}_* \nu^{(s,\beta)} = \mathbb{P}_{\Pi^{(s,\beta)}}$.

The key rôle in the proof of Theorem 1.11 is played by

Proposition 1.16. *For any $\beta > 0$, $s \in \mathbb{R}$, as $n \to \infty$ we have*

$$\nu^{(s,n,\beta)} \to \nu^{(s,\beta)}$$

weakly in the space $\mathfrak{M}_{\text{fin}}(\Omega_P)$.

Proposition 1.16 will be proved in Section 6, and in Section 7, using Proposition 1.16, combined with Lemma 1.14, we will conclude the proof of the main result, Theorem 1.11.

To establish weak convergence of the measures $\nu^{(s,n,\beta)}$, we first study scaling limits of the radial parts of finite-dimensional projections of infinite Pickrell measures.

1.12. **The radial part of the Pickrell measure.** Following Pickrell, to a matrix $z \in \mathrm{Mat}(n,\mathbb{C})$ assign the collection $(\lambda_1(z),\dots,\lambda_n(z))$ of the eigenvalues of the matrix z^*z arranged in non-increasing order. Introduce a map

$$\mathfrak{rad}_n : \mathrm{Mat}(n,\mathbb{C}) \to \mathbb{R}_+^n$$

by the formula

$$\mathfrak{rad}_n : z \to (\lambda_1(z),\dots,\lambda_n(z))\,. \tag{31}$$

The map (31) naturally extends to a map defined on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ for which we keep the same symbol: in other words, the map $\mathfrak{rad}_n$ assigns to an infinite matrix the array of squares of the singular values of its $n \times n$-corner.

The *radial part* of the Pickrell measure $\mu_n^{(s)}$ is now defined as the pushforward of the measure $\mu_n^{(s)}$ under the map $\mathfrak{rad}_n$. Note that, since finite-dimensional unitary groups are compact, and, by definition, for any s and all sufficiently large n, the measure $\mu_n^{(s)}$ assigns finite weight to compact sets, the pushforward is well-defined, for sufficiently large n, even if the measure $\mu^{(s)}$ is infinite.

Slightly abusing notation, we write dz for the Lebesgue measure $\mathrm{Mat}(n,\mathbb{C})$ and $d\lambda$ for the Lebesgue measure on $\mathbb{R}_+^n$.

For the push-forward of the Lebesgue measure $\mathrm{Leb}^{(n)} = dz$ under the map $\mathfrak{rad}_n$ we now have

$$(\mathfrak{rad}_n)_*(dz) = const(n) \cdot \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda,$$

where $const(n)$ is a positive constant depending only on n.

The radial part of the measure $\mu_n^{(s)}$ now takes the form:

$$(\mathfrak{rad}_n)_* \mu_n^{(s)} = const(n, s) \cdot \prod_{i<j} (\lambda_i - \lambda_j)^2 \cdot \frac{1}{(1+\lambda_i)^{2n+s}} d\lambda, \tag{32}$$

where $const(n, s)$ for a positive constant depending on n and s (the constant may change from one formula to another).

Following Pickrell, introduce new variables $u_1, \dots, u_n$ by the formula

$$u_i = \frac{x_i - 1}{x_i + 1}. \tag{33}$$

Proposition 1.17. *In the coordinates (33) the radial part $(\mathfrak{rad}_n)_* \mu_n^{(s)}$ of the measure $\mu_n^{(s)}$ is defined on the cube $[-1, 1]^n$ by the formula*

$$(\mathfrak{rad}_n)_* \mu_n^{(s)} = const(n, s) \cdot \prod_{i<j} (u_i - u_j)^2 \cdot \prod_{i=1}^{n} (1 - u_i)^s \, du_i. \tag{34}$$

In the case $s > -1$, the constant $const(n, s)$ can be chosen in such a way that the right-hand side be a probability measure; in the case $s \le -1$, there is no canonical normalization, the left hand side is defined up to proportionality, and a positive constant can be chosen arbitrarily.

For $s > -1$, Proposition 1.17 yields a determinantal representation for the radial part of the Pickrell measure: namely, the radial part is identified with the Jacobi orthogonal polynomial ensemble in the coordinates (33). Passing to the scaling limit, one obtains the Bessel point process (subject to the change of variable $y = 4/x$).

Similarly, it will develop that for $s \le -1$, the scaling limit of the measures (34) is precise the modified infinite Bessel point process introduced above. Furthermore, if one multiplies the measures (34) by the density $\exp(-\beta S(X)/n^2)$, then the resulting measures are finite and determinantal, and their weak limit, after approproate scaling, is precisely the determinantal measure $\mathbb{P}_{\Pi^{(s,\beta)}}$ of (18). This weak convergence is a key step in the proof of Proposition 1.16.

The study of the case $s \le -1$ thus requires a new object: infinite determinantal measures on spaces of configurations. In the next Section, we proceed to the general construction and description of the properties of infinite determinantal measures.

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2. CONSTRUCTION AND PROPERTIES OF INFINITE DETERMINANTAL MEASURES

2.1. **Preliminary remarks on sigma-finite measures.**

Let Y be a Borel space, and consider a representation

$$Y = \bigcup_{n=1}^{\infty} Y_n$$

of Y as a countable union of an increasing sequence of subsets Y_n, $Y_n \subset Y_{n+1}$. As before, given a measure μ on Y and a subset $Y' \subset Y$, we write $\mu|_{Y'}$ for the restriction of μ onto Y'. Assume that for every n we are given a probability measure $\mathbb{P}_n$ on Y_n. The following proposition is clear.

Proposition 2.1. *A sigma-finite measure $\mathbb{B}$ on Y such that*

$$\frac{\mathbb{B}|_{Y_n}}{\mathbb{B}(Y_n)} = \mathbb{P}_n \tag{35}$$

exists if and only if for any N, n, $N > n$, we have

$$\frac{\mathbb{P}_N|_{Y_n}}{\mathbb{P}_N(Y_n)} = \mathbb{P}_n.$$

The condition 35 *determines the measure $\mathbb{B}$ uniquely up to multiplication by a constant.*

Corollary 2.2. *If $\mathbb{B}_1$, $\mathbb{B}_2$ are two sigma-finite measures on Y such that for all $n \in \mathbb{N}$ we have*

$$0 < \mathbb{B}_1(Y_n) < +\infty, 0 < \mathbb{B}_2(Y_n) < +\infty,$$

and

$$\frac{\mathbb{B}_1|_{Y_n}}{\mathbb{B}_1(Y_n)} = \frac{\mathbb{B}_2|_{Y_n}}{\mathbb{B}_2(Y_n)},$$

then there exists a positive constant $C > 0$ such that $\mathbb{B}_1 = C\mathbb{B}_2$.

2.2. **The unique extension property.**

2.2.1. *Extension from a subset.* Let E be a standard Borel space, let μ be a sigma-finite measure on E, let L be a closed subspace of $L_2(E,\mu)$, let Π be the operator of orthogonal projection onto L, and let $E_0 \subset E$ be a Borel subset. We shall say that the subspace L has the *unique extension property* from E_0 if a function $\varphi \in L$ satisfying $\chi_{E_0}\varphi = 0$ must be the zero function and the subspace $\chi_{E_0}L$ is closed. In general, if a function $\varphi \in L$ satisfying $\chi_{E_0}\varphi = 0$ must be the zero function, then the restricted subspace $\chi_{E_0}L$ still need not be closed: nonetheless, we have the following clear corollary of the open mapping theorem.

Proposition 2.3. *Assume that the closed subspace L is such that a function $\varphi \in L$ satisfying $\chi_{E_0}\varphi = 0$ must be the zero function. The subspace $\chi_{E_0}L$ is closed if and only if there exists $\varepsilon > 0$ such that for any $\varphi \in L$ we have*

$$||\chi_{E\setminus E_0}\varphi|| \leq (1-\varepsilon)||\varphi||, \tag{36}$$

in which case the natural restriction map $\varphi \to \chi_{E_0}\varphi$ is an isomorphism of Hilbert spaces. If the operator $\chi_{E\setminus E_0}\Pi$ is compact, then the condition (36) holds.

Remark. In particular, the condition (36) a fortiori holds if the operator $\chi_{E\setminus E_0}\Pi$ is Hilbert-Schmidt or, equivalently, if the operator $\chi_{E\setminus E_0}\Pi\chi_{E\setminus E_0}$ belongs to the trace class.

The following corollaries are immediate.

Corollary 2.4. *Let g be a bounded nonnegative Borel function on E such that*

$$\inf_{x\in E_0} g(x) > 0. \tag{37}$$

If (36) holds then the subspace $\sqrt{g}L$ is closed in $L_2(E,\mu)$.

Remark. The apparently superfluous square root is put here to keep notation consistent with the remainder of the paper.

Corollary 2.5. *Under the assumptions of Proposition* 2.3*, if (*36*) holds and a Borel function* $g \to [0,1]$ *satisfies (*37*), then the operator* Π^g *of orthogonal projection onto the subspace* $\sqrt{g}L$ *is given by the formula*

$$\Pi^g = \sqrt{g}\Pi(1+(g-1)\Pi)^{-1}\sqrt{g} = \sqrt{g}\Pi(1+(g-1)\Pi)^{-1}\Pi\sqrt{g}. \tag{38}$$

In particular, the operator Π^{E_0} *of orthogonal projection onto the subspace* $\chi_{E_0}L$ *has the form*

$$\Pi^{E_0} = \chi_{E_0}\Pi(1-\chi_{E\setminus E_0}\Pi)^{-1}\chi_{E_0} = \chi_{E_0}\Pi(1-\chi_{E\setminus E_0}\Pi)^{-1}\Pi\chi_{E_0}. \tag{39}$$

Corollary 2.6. *Under the assumptions of Proposition* 2.3*, if (*36*) holds, then, for any subset* $Y \subset E_0$*, once the operator* $\chi_Y\Pi^{E_0}\chi_Y$ *belongs to the trace class, it follows that so does the operator* $\chi_Y\Pi\chi_Y$*, and we have*

$$\mathrm{tr}\chi_Y\Pi^{E_0}\chi_Y \geq \mathrm{tr}\chi_Y\Pi\chi_Y$$

Indeed, from (39) it is clear that if the operator $\chi_Y\Pi^{E_0}$ is Hilbert-Schmidt, then the operator $\chi_Y\Pi$ is also Hilbert-Schmidt. The inequality between traces is also immediate from (39).

2.2.2. *Examples: the Bessel kernel and the modified Bessel kernel.*

Proposition 2.7. (1) *For any* $\varepsilon > 0$*, the operator* $\tilde{J}_s$ *has the unique extension property from the subset* $(\varepsilon, +\infty)$*;*

(2) *For any* $R > 0$*, the operator* $J^{(s)}$ *has the unique extension property from the subset* $(0, R)$*.*

Proof. The first statement is an immediate corollary of the uncertainty principle for the Hankel transform: a function and its Hankel transform cannot both have support of finite measure [16], [17]. (note here that the uncertainty principle is only formulated for $s > -1/2$ in [16] but the more general uncertainty principle of [17] is directly applicable also to the case $s \in [-1, 1/2]$) and the following estimate, which, by definition, is clearly valid for any $R > 0$:

$$\int_0^R \tilde{J}_s(y,y)dy < +\infty.$$

The second statement follows from the first by the change of variable $y = 4/x$. The proposition is proved completely.

2.3. **Inductively determinantal measures.** Let E be a locally compact complete metric space, and let $\mathrm{Conf}(E)$ be the space of configurations on E endowed with the natural Borel structure (see, e.g., [21], [41] and the Appendix).

Given a Borel subset $E' \subset E$, we let $\mathrm{Conf}(E, E')$ be the subspace of configurations all whose particles lie in E'.

Given a measure $\mathbb{B}$ on a set X and a measurable subset $Y \subset X$ such that $0 < \mathbb{B}(Y) < +\infty$, we let $\mathbb{B}|_Y$ stand for the restriction of the measure $\mathbb{B}$ onto the subset Y.

Let μ be a σ-finite Borel measure on E.

We let $E_0 \subset E$ be a Borel subset and assume that for any bounded Borel subset $B \subset E \backslash E_0$ we are given a closed subspace $L^{E_0 \cup B} \subset L_2(E, \mu)$ such that the corresponding projection operator $\Pi^{E_0 \cup B}$ belongs to the space $\mathscr{I}_{1,\mathrm{loc}}(E, \mu)$. We furthermore make the following

Assumption 1. (1) $\left\|\chi_B \Pi^{E_0 \cup B}\right\| < 1\,,\ \chi_B \Pi^{E_0 \cup B} \chi_B \in \mathscr{I}_1(E, \mu)$

(2) *for any subsets $B^{(1)} \subset B^{(2)} \subset E \backslash E_0$, we have*

$$\chi_{E_0 \bigcup B^{(1)}} L^{E_0 \cup B^{(2)}} = L^{E_0 \cup B^{(1)}}.$$

Proposition 2.8. *Under these assumptions, there exists a σ-finite measure $\mathbb{B}$ on $\mathrm{Conf}(E)$ such that*

(1) *for $\mathbb{B}$-almost every configuration, only finitely many of its particles may lie in $E \backslash E_0$;*
(2) *for any bounded Borel subset $B \subset E \backslash E_0$, we have*

$$0 < \mathbb{B}(\mathrm{Conf}\,(E;\ E_0 \cup B)) < +\infty \ \ \textit{and}$$

$$\frac{\mathbb{B}\left|_{\mathrm{Conf}(E;\, E_0 \bigcup B)}\right.}{\mathbb{B}\left(\mathrm{Conf}\,(E;\ E_0 \bigcup B)\right)} = \mathbb{P}_{\Pi^{E_0 \cup B}}\ .$$

Such a measure will be called an *inductively determinantal measure*.

Proposition 2.8 is immediate from Proposition 2.1 combined with Proposition 9.3 and Corollary 9.5 from the Appendix. Note that conditions 1 and 2 define our measure uniquely up to multiplication by a constant.

We now give a sufficient condition for an inductively determinantal measure to be an actual finite determinantal measure.

Proposition 2.9. *Consider a family of projections $\Pi^{E_0 \cup B}$ satisfying the Assumption 1 and the corresponding inductively determinantal measure $\mathbb{B}$. If there exists $R > 0$, $\varepsilon > 0$ such that for all bounded Borel subset $B \subset E \backslash E_0$ we have*

(1) $\left\|\chi_B \Pi^{E_0 \cup B}\right\| < 1 - \varepsilon$*;*
(2) $\mathrm{tr} \chi_B \Pi^{E_0 \cup B} \chi_B < R$*.*

then there exists a projection operator $\Pi \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ onto a closed subspace $L \subset L_2(E, \mu)$ such that

(1) $L^{E_0 \cup B} = \chi_{E_0 \bigcup B} L$ *for all B;*
(2) $\chi_{E \backslash E_0} \Pi \chi_{E \backslash E_0} \in \mathscr{I}_1(E, \mu)$*;*
(3) *the measures $\mathbb{B}$ and $\mathbb{P}_\Pi$ coincide up to multiplication by a constant.*

Proof. By our assumptions, for every bounded Borel subset $B \subset E \backslash E_0$ we are given a closed subspace $L^{E_0 \cup B}$, the range of the operator $\Pi^{E_0 \bigcup B}$, which has the property of unique extension from E_0. The uniform estimate on the norms of the operators $\chi_B \Pi^{E_0 \bigcup B}$ implies the existence of a closed subspace L such that $L^{E_0 \cup B} = \chi_{E_0 \cup B} L$. Now, by our assumptions, the projection operator $\Pi^{E_0 \bigcup B}$ belongs to the space $\mathscr{I}_{1,\mathrm{loc}}(E, \mu)$, whence, for any bounded subset $Y \subset E$, we have

$$\chi_Y \Pi^{E_0 \cup Y} \chi_Y \in \mathscr{I}_1(E, \mu),$$

whence, by Corollary 2.6 applied to the subset $E_0 \cup Y$, it follows that

$$\chi_Y \Pi \chi_Y \in \mathscr{I}_1(E, \mu).$$

It follows that the operator Π of orthogonal projection on L is locally of trace class and therefore induces a unique determinantal probability measure $\mathbb{P}_\Pi$ on $\mathrm{Conf}(E)$. Applying Corollary 2.6 again, we have

$$\mathrm{tr} \chi_{E \backslash E_0} \Pi \chi_{E \backslash E_0} \leq R,$$

and the proposition is proved completely

We now give sufficient conditions for the measure $\mathbb{B}$ to be infinite.

Proposition 2.10. *Make either of the two assumptions:*

(1) *for any $\varepsilon > 0$, there exists a bounded Borel subset $B \subset E \backslash E_0$ such that*
$$||\chi_B \Pi^{E_0 \bigcup B}|| > 1 - \varepsilon$$
(2) *for any $R > 0$, there exists a bounded Borel subset $B \subset E \backslash E_0$ such that*
$$\mathrm{tr} \chi_B \Pi^{E_0 \bigcup B} \chi_B > R\,.$$

Then the measure $\mathbb{B}$ is infinite.

Proof. Recall that we have

$$\frac{\mathbb{B}\,(\mathrm{Conf}\,(E;\, E_0))}{\mathbb{B}\,(\mathrm{Conf}\,(E;\, E_0 \bigcup B))} = \mathbb{P}_{\Pi^{E_0 \bigcup B}}\,(\mathrm{Conf}\,(E;\, E_0)) = \\ = \det\left(1 - \chi_B \Pi^{E_0 \bigcup B} \chi_B\right). \tag{40}$$

Under the first assumption, it is immediate that the top eigenvalue of the self-adjoint trace-class operator $\chi_B \Pi^{E_0 \bigcup B} \chi_B$ exceeds $1 - \varepsilon$, whence

$$\det\left(1 - \chi_B \Pi^{E_0 \bigcup B} \chi_B\right) \leq \varepsilon.$$

Under the second assumption, write

$$\det\left(1 - \chi_B \Pi^{E_0 \bigcup B} \chi_B\right) \leqslant \exp\left(-\mathrm{tr} \chi_B \Pi^{E_0 \bigcup B} \chi_B\right) \leq \exp(-R). \tag{41}$$

In both cases, the ratio

$$\frac{\mathbb{B}\left(\mathrm{Conf}\left(E;\ E_0\right)\right)}{\mathbb{B}\left(\mathrm{Conf}\left(E;\ E_0 \bigcup B\right)\right)}$$

can be made arbitrary small by an appropriate choice of B, which implies that the measure $\mathbb{B}$ is infinite. The proposition is proved.

2.4. **General construction of infinite determinantal measures .** By the Macchì-Soshnikov Theorem, under some additional assumptions, a determinantal measure can be assigned to an operator of orthogonal projection, or, in other words, to a closed subspace of $L_2(E,\mu)$. In a similar way, an infinite determinantal measure will be assigned to a subspace H of *locally* square-integrable functions.

Recall that $L_{2,\mathrm{loc}}(E,\mu)$ is the space of all measurable functions $f : E \to \mathbb{C}$ such that for any bounded subset $B \subset E$ we have

$$\int\limits_B |f|^2 d\mu < +\infty. \tag{42}$$

Choosing an exhausting family B_n of bounded sets (for instance, balls with fixed centre and of radius tending to infinity) and using (42) with $B = B_n$, we endow the space $L_{2,\mathrm{loc}}(E,\mu)$ with a countable family of seminorms which turns it into a complete separable metric space; the topology thus defined does not, of course, depend on the specific choice of the exhausting family.

Let $H \subset L_{2,\mathrm{loc}}(E,\mu)$ be a linear subspace. If $E' \subset E$ is a Borel subset such that $\chi_{E'}H$ is a closed subspace of $L_2(E,\mu)$, then we denote by $\Pi^{E'}$ the operator of orthogonal projection onto the subspace $\chi_{E'}H \subset L_2(E,\mu)$. We now fix a Borel subset $E_0 \subset E$; informally, E_0 is the set where the particles accumulate. We impose the following assumption on E_0 and H.

Assumption 2. (1) *For any bounded Borel set $B \subset E$, the space $\chi_{E_0 \cup B}H$ is a closed subspace of $L_2(E,\mu)$;*

(2) *For any bounded Borel set $B \subset E \setminus E_0$, we have*

$$\Pi^{E_0 \cup B} \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu), \quad \chi_B \Pi^{E_0 \cup B} \chi_B \in \mathscr{I}_1(E,\mu); \tag{43}$$

(3) *If $\varphi \in H$ satisfies $\chi_{E_0}\varphi = 0$, then $\varphi = 0$.*

If a subspace H and the subset E_0 have the property that any $\varphi \in H$ satisfying $\chi_{E_0}\varphi = 0$ must be the zero function, then we shall say that H has the *property of unique extension* from E_0.

Theorem 2.11. *Let E be a locally compact complete metric space, and let μ be a σ-finite Borel measure on E. If a subspace $H \subset L_{2,\mathrm{loc}}(E,\mu)$ and a Borel subset $E_0 \subset E$ satisfy Assumption 2, then there exists a σ-finite Borel measure $\mathbb{B}$ on $\mathrm{Conf}(E)$ such that*

(1) $\mathbb{B}$-*almost every configuration has at most finitely many particles outside of* E_0*;*
(2) *for any bounded Borel (possibly empty) subset* $B \subset E \setminus E_0$ *we have* $0 < \mathbb{B}(\mathrm{Conf}(E; E_0 \cup B)) < +\infty$ *and*

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E;E_0\cup B)}}{\mathbb{B}(\mathrm{Conf}(E; E_0 \cup B))} = \mathbb{P}_{\Pi^{E_0\cup B}}.$$

The requirements (1) and (2) determine the measure $\mathbb{B}$ *uniquely up to multiplication by a positive constant.*

We denote $\mathbf{B}(H, E_0)$ the one-dimensional cone of nonzero infinite determinantal measures induced by H and E_0, and, slightly abusing notation, we write $\mathbb{B} = \mathbb{B}(H, E_0)$ for a representative of the cone.

Remark. If B is a bounded set, then, by definition, we have

$$\mathbf{B}(H, E_0) = \mathbf{B}(H, E_0 \cup B).$$

Remark. If $E' \subset E$ is a Borel subset such that $\chi_{E_0\cup E'}$ is a closed subspace in $L_2(E, \mu)$ and the operator $\Pi^{E_0\cup E'}$ of orthogonal projection onto the subspace $\chi_{E_0\cup E'}H$ satisfies

$$\Pi^{E_0\cup E'} \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu), \quad \chi_{E'}\Pi^{E_0\cup E'}\chi_{E'} \in \mathscr{I}_1(E, \mu), \tag{44}$$

then, exhausting E' by bounded sets, from Theorem 2.11 one easily obtains $0 < \mathbb{B}(\mathrm{Conf}(E; E_0 \cup E')) < +\infty$ and

$$\frac{\mathbb{B}|_{\mathrm{Conf}(E;E_0\cup E')}}{\mathbb{B}(\mathrm{Conf}(E; E_0 \cup E'))} = \mathbb{P}_{\Pi^{E_0\cup E'}}.$$

2.5. **Change of variables for infinite determinantal measures.** Let $F : E \to E$ be a homeomorphism. The homeomorphism F induces a homeomorphism of the space $\mathrm{Conf}(E)$, for which, slightly abusing notation, we keep the same symbol: given $X \in \mathrm{Conf}(E)$, the particles of the configuration $F(X)$ have the form $F(x)$ over all $x \in X$.

Assume now that the measures $F_*\mu$ and μ are equivalent, and let $\mathbb{B} = \mathbb{B}(H, E_0)$ be an infinite determinantal measure. Introduce the subspace

$$F^*H = \{\varphi(F(x)) \cdot \sqrt{\frac{dF_*\mu}{d\mu}}, \varphi \in H\}.$$

From the definitions we now clearly have the following

Proposition 2.12. *The push-forward of the infinite determinantal measure* $\mathbb{B} = \mathbb{B}(H, E_0)$ *has the form*

$$F_*\mathbb{B} = \mathbb{B}(F^*H, F(E_0)).$$

2.6. **Example: infinite orthogonal polynomial ensembles.** Let ρ be a nonnegative function on $\mathbb{R}$ not identically equal to zero. Take $N \in \mathbb{N}$ and endow the set $\mathbb{R}^N$ with the measure

$$\prod_{1\leqslant i,j\leqslant N} (x_i - x_j)^2 \prod_{i=1}^{N} \rho(x_i) dx_i. \tag{45}$$

If for $k = 0, \dots, 2N-2$ we have

$$\int_{-\infty}^{+\infty} x^k \rho(x) dx < +\infty,$$

then the measure (45) has finite mass and, after normalization, yields a determinantal point process on $\mathrm{Conf}(\mathbb{R})$.

Given a finite family of functions $f_1, \dots, f_N$ on the real line, let $\mathrm{span}(f_1, \dots, f_N)$ stand for the vector space these functions span. For a general function ρ, introduce the subspace $H(\rho) \subset L_{2,\mathrm{loc}}(\mathbb{R}, \mathrm{Leb})$ by the formula

$$H(\rho) = \mathrm{span}\left(\sqrt{\rho(x)}, x\sqrt{\rho(x)}, \dots, x^{N-1}\sqrt{\rho(x)}\right).$$

The measure (45) is an infinite determinantal measure, as is shown by the following immediate

Proposition 2.13. *Let ρ be a positive continuous function on $\mathbb{R}$, and let $(a, b) \subset \mathbb{R}$ be a nonempty interval such that the function ρ is positive and bounded in restriction to (a, b). Then the measure (45) is an infinite determinantal measure of the form $\mathbb{B}(H(\rho), (a, b))$.*

2.7. **Multiplicative functionals of infinite determinantal measures.** Our next aim is to show that, under some additional assumptions, an infinite determinantal measure can be represented as a product of a finite determinantal measure and a multiplicative functional.

Proposition 2.14. *Let a subspace $H \subset L_{2,\mathrm{loc}}(E, \mu)$ and a Borel subset E_0 induce an infinite determinantal measure $\mathbb{B} = \mathbb{B}(H, E_0)$. Let $g\colon E \to (0, 1]$ be a positive Borel function such that $\sqrt{g}H$ is a closed subspace in $L_2(E, \mu)$, and let Π^g be the corresponding projection operator. Assume additionally*

(1) $\sqrt{1-g}\,\Pi^{E_0}\sqrt{1-g} \in \mathscr{I}_1(E, \mu)$ *;*
(2) $\chi_{E\setminus E_0}\Pi^g \chi_{E\setminus E_0}\mathscr{I}_1(E, \mu)$ *;*
(3) $\Pi^g \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$

Then the multiplicative functional Ψ_g is $\mathbb{B}$-almost surely positive, $\mathbb{B}$-integrable, and we have

$$\frac{\Psi_g \mathbb{B}}{\int\limits_{\mathrm{Conf}(E)} \Psi_g \, d\mathbb{B}} = \mathbb{P}_{\Pi^g} \ .$$

Before starting the proof, we prove some auxiliary propositions.
First, we note a simple corollary of unique extension property.

Proposition 2.15. . *Let $H \subset L_{2,\mathrm{loc}}(E,\mu)$ have the property of unique extension from E_0, and let $\psi \in L_{2,\mathrm{loc}}(E,\mu)$ be such that $\chi_{E_0 \bigcup B}\psi \in \chi_{E_0 \bigcup B} H$ for any bounded Borel set $B \subset E \backslash E_0$. Then $\psi \in H$.*

Proof. Indeed, for any B there exists $\psi_B \in L_{2,\mathrm{loc}}(E,\mu)$ such that $\chi_{E_0 \bigcup B}\psi_B = \chi_{E_0 \bigcup B}\psi$. Take two bounded Borel sets B_1 and B_2 and note that $\chi_{E_0}\psi_{B_1} = \chi_{E_0}\psi_{B_2} = \chi_{E_0}\psi$, whence, by the unique extension property, $\psi_{B_1} = \psi_{B_2}$. Thus all the functions ψ_B coincide and also coinside with ψ, which, consequently, belongs to H.

Our next proposition gives a sufficient condition for a subspace of locally square-integrable functions to be a closed subspace in L_2.

Proposition 2.16. *Let $L \subset L_{2,\mathrm{loc}}(E,\mu)$ be a subspace such that*

(1) *for any bounded Borel $B \subset E \backslash E_0$ the space $\chi_{E_0 \bigcup B} L$ is a closed subspace of $L_2(E,\mu)$;*
(2) *the natural restriction map $\chi_{E_0 \bigcup B} L \to \chi_{E_0} L$ is an isomorphism of Hilbert spaces, and the norm of its inverse is bounded above by a positive constant independent of B.*

Then L is a closed subspace of $L_2(E,\mu)$, and the natural restriction map $L \to \chi_{E_0} L$ is an isomorphism of Hilbert spaces.

Proof. If L contained a function with non-integrable square, then for an appropriately chosen of B the inverse of the restriction isomorphism $\chi_{E_0 \bigcup B} L \to \chi_{E_0} L$ would have an arbitrarily large norm. That L is closed follows from the unique extension property and Proposition 2.15.

We now proceed with the proof of Proposition 2.14.

First we check that for any bounded Borel $B \subset E \backslash E_0$ we have

$$\sqrt{1-g}\,\Pi^{E_0 \bigcup B}\sqrt{1-g} \in \mathscr{I}_1(E,\mu) \tag{46}$$

Indeed, the definition of an infinite determinantal measure implies

$$\chi_B \Pi^{E_0 \bigcup B} \in \mathscr{I}_2(E,\mu),$$

whence, a fortiori, we have

$$\sqrt{1-g}\,\chi_B \Pi^{E_0 \bigcup B} \in \mathscr{I}_2(E,\mu).$$

Now recall that
$$\Pi^{E_0} = \chi_{E_0}\Pi^{E_0\bigcup B}\left(1-\chi_B\Pi^{E_0\bigcup B}\right)^{-1}\Pi^{E_0\bigcup B}\chi_{E_0}.$$
The relation
$$\sqrt{1-g}\Pi^{E_0}\sqrt{1-g}\in\mathscr{I}_1(E,\mu)$$
therefore implies
$$\sqrt{1-g}\chi_{E_0}\Pi^{E_0\bigcup B}\chi_{E_0}\sqrt{1-g}\in\mathscr{I}_1(E,\mu),$$
or, equivalently,
$$\sqrt{1-g}\chi_{E_0}\Pi^{E_0\bigcup B}\in\mathscr{I}_2(E,\mu).$$
We coincide that
$$\sqrt{1-g}\Pi^{E_0\bigcup B}\in\mathscr{I}_2(E,\mu),$$
or, equivalently, that
$$\sqrt{1-g}\Pi^{E_0\bigcup B}\sqrt{1-g}\in\mathscr{I}_1(E,\mu)$$
as desired.

We next check that the subspace $\sqrt{g}H\chi_{E_0\bigcup B}$ is closed in $L_2(E,\mu)$. But this is immediate from closedness of the subspace $\sqrt{g}H$, the unique extension property from the subset E_0, which the subspace $\sqrt{g}H$ has, since so does H, and our assumption
$$\chi_{E\setminus E_0}\Pi^g\chi_{E\setminus E_0}\in\mathscr{I}_1(E,\mu).$$

We now let $\Pi^{g\chi_{E_0\cup B}}$ be the operator of orthogonal projection onto the subspace $\sqrt{g}H\chi_{E_0\bigcup B}$.

It follows from the above that for any bounded Borel set $B\subset E\setminus E_0$ the multiplicative functional Ψ_g is $\mathbb{P}_{\Pi^{E_0\cup B}}$-almost surely positive and, furthermore, that we have
$$\frac{\Psi_g\mathbb{P}_{\Pi^{E_0\cup B}}}{\displaystyle\int\Psi_g\,d\mathbb{P}_{\Pi^{E_0\cup B}}}=\mathbb{P}_{\Pi^{g\chi_{E_0\cup B}}}\ ,$$
where $\Pi^{g\chi_{E_0\cup B}}$ is the operator of orthogonal projection onto the closed subspace $\sqrt{g}\chi_{E_0\bigcup B}H$.

It follows now that for any bounded Borel $B\subset E\setminus E_0$ we have
$$\frac{\Psi_{g\chi_{E_0\cup B}}\mathbb{B}}{\displaystyle\int\Psi_{g\chi_{E_0\cup B}}\,d\mathbb{B}}=\mathbb{P}_{\Pi^{g\chi_{E_0\cup B}}}\ . \tag{47}$$
It remains to note that (47) immediately implies the statement of Proposition 2.14, whose proof is thus complete.

2.8. **Infinite determinantal measures obtained as finite-rank perturbations of determinantal probability measures.**

2.8.1. *Construction of finite-rank perturbations.* We now consider infinite determinantal measures induced by subspaces H obtained by adding a finite-dimensional subspace V to a closed subspace $L \subset L_2(E, \mu)$.

Let, therefore, $Q \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ be the operator of orthogonal projection onto a closed subspace $L \subset L_2(E, \mu)$, let V be a finite-dimensional subspace of $L_{2,\mathrm{loc}}(E, \mu)$ such that $V \cap L_2(E, \mu) = 0$, and set $H = L + V$. Let $E_0 \subset E$ be a Borel subset. We shall need the following assumption on L, V and E_0.

Assumption 3. (1) $\chi_{E \setminus E_0} Q \chi_{E \setminus E_0} \in \mathscr{I}_1(E, \mu)$;
(2) $\chi_{E_0} V \subset L_2(E, \mu)$;
(3) *if* $\varphi \in V$ *satisfies* $\chi_{E_0}\varphi \in \chi_{E_0} L$, *then* $\varphi = 0$;
(4) *if* $\varphi \in L$ *satisfies* $\chi_{E_0}\varphi = 0$, *then* $\varphi = 0$.

Proposition 2.17. *If* L, V *and* E_0 *satisfy Assumption* 3 *then the subspace* $H = L + V$ *and* E_0 *satisfy Assumption* 2.

In particular, for any bounded Borel subset B, the subspace $\chi_{E_0 \cup B} L$ is closed, as one sees by taking $E' = E_0 \cup B$ in the following clear

Proposition 2.18. *Let* $Q \in \mathscr{I}_{1,\mathrm{loc}}(E, \mu)$ *be the operator of orthogonal projection onto a closed subspace* $L \in L_2(E, \mu)$. *Let* $E' \subset E$ *be a Borel subset such that* $\chi_{E'} Q \chi_{E'} \in \mathscr{I}_1(E, \mu)$ *and that for any function* $\varphi \in L$, *the equality* $\chi_{E'}\varphi = 0$ *implies* $\varphi = 0$. *Then the subspace* $\chi_{E'} L$ *is closed in* $L_2(E, \mu)$.

The subspace H and the Borel subset E_0 therefore define an infinite determinantal measure $\mathbb{B} = \mathbb{B}(H, E_0)$. The measure $\mathbb{B}(H, E_0)$ is indeed infinite by Proposition 2.10.

2.8.2. *Multiplicative functionals of finite-rank perturbations.* Proposition 2.14 now has the following immediate

Corollary 2.19. *Let* L, V *and* E_0 *induce an infinite determinantal measure* $\mathbb{B}$. *Let* $g\colon E \to (0, 1]$ *be a positive measurable function. If*
(1) $\sqrt{g}V \subset L_2(E, \mu)$;
(2) $\sqrt{1-g}\,\Pi\sqrt{1-g} \in \mathscr{I}_1(E, \mu)$,

then the multiplicative functional Ψ_g *is* $\mathbb{B}$*-almost surely positive and integrable with respect to* $\mathbb{B}$, *and we have*

$$\frac{\Psi_g \mathbb{B}}{\displaystyle\int \Psi_g \, d\mathbb{B}} = \mathbb{P}_{\Pi^g},$$

where Π^g *is the operator of orthogonal projection onto the closed subspace* $\sqrt{g}L + \sqrt{g}V$.

2.9. **Example: the infinite Bessel point process.** We are now ready to prove Proposition 1.1 on the existence of the infinite Bessel point process $\widetilde{\mathbb{B}}^{(s)}$, $s \leqslant -1$. We first need the following property of the usual Bessel point process $\widetilde{J}_s$, $s > -1$. As before, let $\widetilde{L}_s$ be the range of the projection operator $\widetilde{J}_s$.

Lemma 2.20. *Let $s > -1$ be arbitrary. Then*

(1) *For any $R > 0$ the subspace $\chi_{(R,+\infty)}\widetilde{L}_s$ is closed in $L_2\big((0,+\infty),\mathrm{Leb}\big)$, and the corresponding projection operator $\widetilde{J}_{s,R}$ is locally of trace class;*
(2) *For any $R > 0$ we have*
$$\mathbb{P}_{\widetilde{J}_s}\left(\mathrm{Conf}\left((0,+\infty),(R,+\infty)\right)\right) > 0,$$
and
$$\frac{\mathbb{P}_{\widetilde{J}_s}\big|_{\mathrm{Conf}((0,+\infty),(R,+\infty))}}{\mathbb{P}_{\widetilde{J}_s}\left(\mathrm{Conf}\left((0,+\infty),(R,+\infty)\right)\right)} = \mathbb{P}_{\widetilde{J}_{s,R}}\,.$$

Proof. First, for any $R > 0$ we clearly have
$$\int\limits_0^R \widetilde{J}_s(x,x)\,dx < +\infty$$
or, equivalently,
$$\chi_{(0,R)}\,\widetilde{J}_s\,\chi_{(0,R)} \in \mathscr{I}_1\big((0,+\infty),\mathrm{Leb}\big).$$

The Lemma follows now from the unique extension property of the Bessel point process. The Lemma is proved completely. □

Now let $s \leqslant -1$ and recall that $n_s \in \mathbb{N}$ is defined by the relation
$$\frac{s}{2} + n_s \in \left(-\frac{1}{2},\frac{1}{2}\right].$$

Let
$$\check{V}^{(s)} = \mathrm{span}\left(y^{s/2}, x^{s/2+1},\ldots,\frac{J_{s+2n_s-1}\left(\sqrt{y}\right)}{\sqrt{y}}\right).$$

Proposition 2.21. *We have $\dim\check{V}^{(s)} = n_s$ and for any $R > 0$ we have*
$$\chi_{(0,R)}\widetilde{V}^{(s)}\bigcap L_2\big((0,+\infty),\mathrm{Leb}\big) = 0.$$

In other words, if a linear combination
$$\Phi^{(\alpha)} = \alpha_0\chi_{(0,R)}\frac{J_{s+2n_s-1}\big(\sqrt{y}\big)}{\sqrt{y}} + \sum_{i=1}^{2n_s-2}\alpha_i\chi_{(0,R)}y^{s/2+i}$$

lies in L_2, then in fact, all the coefficients are zero: $\alpha_0 = \cdots = 0$.

First assume that not all coefficients $\alpha_1, \ldots, \alpha_{2n_s-2}$ are zero. Let $i > 0$ be the smallest index such that $\alpha_i \neq 0$. But then

$$\lim_{y \to 0} \Phi^{(\alpha)}(y) y^{-s/2-i} = \alpha_i \neq 0,$$

and a function with asymptotics $y^{s/2+i}$ at zero cannot be square-integrable. It remains to consider the case when only $\alpha_0 \neq 0$: but the function

$$\frac{J_{s+2n_s-1}(\sqrt{y})}{\sqrt{y}},$$

by definition, fails to be square-integrable in any nonempty interval $(0, R)$. The proposition is proved completely.

Proposition 2.21 immediately implies the existence of the infinite Bessel point process $\tilde{\mathbb{B}}^{(s)}$ and concludes the proof of Proposition 1.1.

Effectuating the change of variable

$$y = 4/x,$$

we also establish the existence of the modified infinite Bessel point process $\mathbb{B}^{(s)}$.

Furthermore, using the characterization of multiplicative functionals of infinite determinantal measures given by Proposition 2.14 and Corollary 2.19 , we arrive at the proof of Propositions 1.5, 1.6, 1.7.

3. Convergence of determinantal measures.

3.1. **Convergence of operators and convergence of measures.** We consider determinantal probability measures induced by positive contractions and start by recalling that convergence of a sequence of such operators in the space of locally trace-class operators implies the weak convergence of corresponding determinantal probability measures in the space of finite measures on the space of configurations.

Proposition 3.1. *Assume that the operators $K_n \in \mathscr{I}_{1,\mathrm{loc}}\big(E, \mu\big)$, $n \in \mathbb{N}$, $K \in \mathscr{I}_{1,\mathrm{loc}}\big(E, \mu\big)$ induce determinantal probability measures $\mathbb{P}_{K_n}$, $n \in \mathbb{N}$, $\mathbb{P}_K$ on $\mathrm{Conf}(E)$. If $K_n \to K$ in $\mathscr{I}_{1,\mathrm{loc}}\big(E, \mu\big)$ as $n \to \infty$, then $\mathbb{P}_{K_n} \to \mathbb{P}_K$ with respect to the weak topology on $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}(E))$ as $n \to \infty$.*

This proposition is immediate from the definition of determinantal probability measures and Proposition 9.1 from the Appendix. From the classical Heine-Mehler asymptotics (cf. Proposition 8.3 in the Appendix) we now have the following immediate

Corollary 3.2. *For any $s > -1$, we have*

$$\tilde{K}_n^{(s)} \to \tilde{J}_s \text{ in } \mathscr{I}_{1,\mathrm{loc}}((0,+\infty), \mathrm{Leb})$$

and

$$\mathbb{P}_{\tilde{K}_n^{(s)}} \to \mathbb{P}_{\tilde{J}_s} \text{ in } \mathfrak{M}_{\mathrm{fin}}\mathrm{Conf}((0,+\infty)).$$

Our next aim is to show that, under certain additional assumptions, the convergence above persists under passage to induced processes as well as to finite-rank perturbations. We proceed to precise statements.

3.2. **Convergence of induced processes.** Recall that if Π is a projection operator acting on $L_2(E,\mu)$ and g is a nonnegative bounded measurable function on E such that the operator $1 + (g-1)\Pi$ is invertible, then we have set

$$\tilde{\mathfrak{B}}(g,\Pi) = \sqrt{g}\Pi(1 + (g-1)\Pi)^{-1}\sqrt{g}.$$

We now fix g and establish the connection between convergence of the sequence Π_n and the corresponding sequence $\tilde{\mathfrak{B}}(g,\Pi_n)$.

Proposition 3.3. *Let $\Pi_n, \Pi \in \mathscr{I}_{1,\mathrm{loc}}$ be orthogonal projection operators, and let $g : E \to [0,1]$ be a measurable function such that*

$$\sqrt{1-g}\Pi\sqrt{1-g} \in \mathscr{I}_1(E,\mu), \sqrt{1-g}\Pi_n\sqrt{1-g} \in \mathscr{I}_1(E,\mu), n \in \mathbb{N}.$$

Assume furthermore that

(1) $\Pi_n \to \Pi$ *in* $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ *as* $n \to \infty$*;*
(2) $\lim\limits_{n\to\infty} \mathrm{tr}\sqrt{1-g}\Pi_n\sqrt{1-g} = \mathrm{tr}\sqrt{1-g}\Pi\sqrt{1-g}$*;*
(3) *the operator* $1 + (g-1)\Pi$ *is invertible.*

Then the operators $1 + (g-1)\Pi_n$ are also invertible for all sufficiently large n, and we have

$$\tilde{\mathfrak{B}}(g,\Pi_n) \to \tilde{\mathfrak{B}}(g,\Pi) \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$$

and, consequently,

$$\mathbb{P}_{\tilde{\mathfrak{B}}(g,\Pi_n)} \to \mathbb{P}_{\tilde{\mathfrak{B}}(g,\Pi)}$$

with respect to the weak topology on $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}(E))$ as $n \to \infty$.

Remark. The second requirement could have been replaced by the requirement that $(g-1)\Pi_n$ converge to $(g-1)\Pi$ in norm, which is weaker and is what we shall actually use; nonetheless, in applications it will be more convenient to check the convergence of traces rather than the norm convergence of operators.

Proof. The first two requirements and Grümm's Theorem (see Simon [40]) imply that

$$\sqrt{1-g}\Pi_n \to \sqrt{1-g}\Pi \text{ in } \mathscr{I}_2(E,\mu),$$

whence, a fortiori,

$$(g-1)\Pi_n \to (g-1)\Pi$$

in norm as $n \to \infty$. We now take a bounded Borel subset $D \subset E$ and check that, as $n \to \infty$, we have

$$\chi_D \tilde{\mathfrak{B}}(g, \Pi_n)\chi_D \to \chi_D \text{ in } \mathscr{I}_1(E, \mu).\tilde{\mathfrak{B}}(g, \Pi)\chi_D \tag{48}$$

Our assumptions directly imply the norm convergence

$$(1 + (g-1)\Pi_n)^{-1} \to (1 + (g-1)Pi)^{-1}. \tag{49}$$

Furthermore,

$$\chi_D \Pi_n \to \chi_D \Pi$$

as $n \to \infty$ in the strong operator topology; besides, by our assumptions, we have

$$\lim_{n\to\infty} \mathrm{tr}\chi_D \Pi_n \chi_D = \mathrm{tr}\chi_D \Pi \chi_D,$$

whence, by Grümm's Theorem , we have $\chi_D \Pi_n \to \chi_D \Pi$ in Hilbert-Schmidt norm, and, a fortiori, in norm.

It follows that the convergence (48) also takes place in norm. To verify the desired $\mathscr{I}_1$ convergence, by Grümm's Theorem again, it suffices to check the relation

$$\lim_{n\to\infty} \mathrm{tr}\chi_D \tilde{\mathfrak{B}}(g, \Pi_n)\chi_D = \mathrm{tr}\chi_D \tilde{\mathfrak{B}}(g, \Pi)\chi_D. \tag{50}$$

First, if A is a bounded operator, and $K_1, K_2 \in \mathscr{I}_2$, then one directly verifies the inequality

$$\mathrm{tr}(K_1^* A K_2) \leq ||K_1||_{\mathscr{I}_2} \cdot ||A|| \cdot ||K_2||_{\mathscr{I}_2}.$$

It easily follows that the function $\mathrm{tr}(K_1^* A K_2)$ is continuous as long as K_1, K_2 are operators in $\mathscr{I}_2$, and A is a bounded operator. The desired convergence of traces (50) follows from the said continuity since

$$\chi_D \tilde{\mathfrak{B}}(g, \Pi)\chi_D = \chi_D \sqrt{g-1}\Pi(1 + (g-1)\Pi)^{-1}\Pi\sqrt{g-1}\chi_D,$$

and we have the norm convergence (49) and the $\mathscr{I}_2$-convergence

$$\chi_D \Pi_n \to \chi_D \Pi.$$

3.2.1. *Convergence of finite-rank perturbations.* We now proceed to the study of convergence of finite-rank perturbations of locally trace-class projection operators. Let L_n, $L \subset L_2(E, \mu)$ be closed subspaces, and let Π_n, Π be the corresponding orthogonal projection operators. Assume we are given non-zero vectors $v^{(n)} \in L_2(E, \mu)$, $n \in \mathbb{N}$, $v \in L_2(E, \mu)$, and let $\widetilde{\Pi}_n, \widetilde{\Pi}$ be the operators of orthogonal projection onto, respectively, the subspaces $L_n + \mathbb{C}v^{(n)}$, $n \in \mathbb{N}$ and $L \oplus \mathbb{C}v$.

Proposition 3.4. *Assume*

(1) $\Pi_n \to \Pi$ *in the strong operator topology as* $n \to \infty$*;*
(2) $v^{(n)} \to v$ *in* $L_2(E,\mu)$ *as* $n \to \infty$*;*
(3) $v \notin L$.

Then $\widetilde{\Pi}_n \to \widetilde{\Pi}$ *in the strong operator topology as* $n \to \infty$.

If, additionally,

$$\Pi_n \to \Pi \textit{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu) \textit{ as } n \to \infty,$$

then also

$$\widetilde{\Pi}_n \to \widetilde{\Pi} \textit{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu) \textit{ as } n \to \infty.$$

Let $\mathrm{angle}(v,H)$ stands for the angle between a vector v and a subspace H. Our assumptions imply that there exists $\alpha_0 > 0$ such that

$$\mathrm{angle}(v_n, L_n) \geq \alpha_0.$$

Decompose

$$v^{(n)} = \beta(n)\widetilde{v}_1^{(n)} + \widehat{v}^{(n)}\,,$$

where $\widetilde{v}^{(n)} \in L_n^{\perp}$, $\|\widetilde{v}^{(n)}\| = 1$, $\widehat{v}^{(n)} \in L_n$. In this case we have

$$\widetilde{\Pi}_n = \Pi_n + P_{\widetilde{v}^{(n)}}\,,$$

where $P_{\widetilde{v}^{(n)}} \colon v \to \langle v, \widetilde{v}^{(n)}\rangle \widetilde{v}^{(n)}$, is the operator of the orthogonal projection onto the subspace $\mathbb{C}\widetilde{v}^{(n)}$.

Similarly, decompose

$$v = \beta\widetilde{v} + \widehat{v}$$

with $\widetilde{v} \in L^{\perp}$, $\|\widetilde{v}\| = 1$, $\widehat{v} \in L$, and, again, write

$$\widetilde{\Pi}_n = \Pi_n + P_{\widetilde{v}_1}\,,$$

with $P_{\widetilde{v}}(v) = \langle v, \widetilde{v}\rangle\widetilde{v}$.

Our assumptions 2 and 3 imply that $\widetilde{v}^{(n)} \to \widetilde{v}$ in $L_2(E,\mu)$. It follows that $P_{\widetilde{v}^{(n)}} \to P_{\widetilde{v}}$ in the strong operator topology and also, since our operators have one-dimensional range, in $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$, which implies the proposition.

The case of perturbations of higher rank follows by induction. Let $m \in \mathbb{N}$ be arbitrary and assume we are given non-zero vectors $v_1^{(n)}, v_2^{(n)}, \dots, v_m^{(n)} \in L_2(E,\mu)$, $n \in \mathbb{N}$, $v_1, v_2, \dots, v_m \in L_2(E,\mu)$. Let

$$\widetilde{L}_n = L_n \oplus \mathbb{C}v_1^{(n)} \oplus \cdots \oplus \mathbb{C}v_m^{(n)}\,,$$

$$\widetilde{L} = L \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{C}v_m\,,$$

and let $\widetilde{\Pi}_n, \widetilde{\Pi}$ be the corresponding projection operators.

Applying Proposition 3.4 inductively, we obtain

Proposition 3.5. *Assume*

(1) $\Pi_n \to \Pi$ *in the strong operator topology as* $n \to \infty$*;*
(2) $v_i^{(n)} \to v_i$ *in* $L_2(E,\mu)$ *as* $n \to \infty$ *for any* $i = 1, \dots, m$*;*

(3) $v_k \notin L \oplus \mathbb{C}v_1 \oplus \cdots \oplus \mathbb{R}v_{k-1}$, $k = 1, \ldots, m$.

Then $\widetilde{\Pi}_n \to \widetilde{\Pi}$ *in the strong operator topology as* $n \to \infty$. *If, additionally,*

$$\Pi_n \to \Pi \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu) \text{ as } n \to \infty,$$

then also

$$\widetilde{\Pi}_n \to \widetilde{\Pi} \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu) \text{ as } n \to \infty,$$

and, consequently, $\mathbb{P}_{\widetilde{\Pi}_n} \to \mathbb{P}_{\widetilde{\Pi}}$ *with respect to the weak topology on* $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}(E))$ *as* $n \to \infty$.

3.3. **Application to infinite determinantal measures.**

Take a sequence

$$\mathbb{B}^{(n)} = \mathbb{B}\left(H^{(n)}, E_0\right)$$

of infinite determinantal measures with $H^{(n)} = L^{(n)} + V^{(n)}$, where $L^{(n)}$ is, as before, the range of a projection operator $\Pi^{(n)} \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$, and $V^{(n)}$ is finite-dimensional. Note that the subset E_0 is fixed throughout.

Our aim is to give sufficient conditions for convergence of $\mathbb{B}^{(n)}$ to a limit measure $\mathbb{B} = \mathbb{B}(H, E_0)$, $H = L + V$, the subspace L being the range of a projection operator $\Pi \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$.

Proposition 3.6. *Assume*

(1) $\Pi^{(n)} \to \Pi$ *in* $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ *as* $n \to \infty$ *;*

(2) *the subspace* $V^{(n)}$ *admits a basis* $v_1^{(n)}, \ldots, v_m^{(n)}$ *and the subspace* V *admits a basis* $v_1, \ldots, v_m$ *such that*

$$v_i^{(n)} \to v_i \text{ in } L_{2,\mathrm{loc}}(E,\mu) \text{ as } n \to \infty \text{ for all } i = 1, \ldots, m\,.$$

Let $g\colon E \to [0,1]$ *be a positive measurable function such that*

(1) $\sqrt{1-g}\,\Pi^{(n)}\sqrt{1-g} \in \mathscr{I}_1(E,\mu)$, $\sqrt{1-g}\,\Pi\sqrt{1-g} \in \mathscr{I}_1(E,\mu)$ *;*

(2) $\lim\limits_{n\to\infty} \mathrm{tr}\sqrt{1-g}\,\Pi^{(n)}\sqrt{1-g} = \mathrm{tr}\sqrt{1-g}\,\Pi\sqrt{1-g}$ *;*

(3) $\sqrt{g}V^{(n)} \subset L_2(E,\mu)$, $\sqrt{g}V \subset L_2(E,\mu)$ *;*

(4) $\sqrt{g}v_i^{(n)} \to \sqrt{g}v_i$ *in* $\subset L_2(E,\mu)$ *as* $n \to \infty$ *for all* $i = 1, \ldots, m$.

Then

(1) *the subspaces* $\sqrt{g}H^{(n)}$ *and* $\sqrt{g}H$ *are closed ;*

(2) *the operators* $\Pi^{(g,n)}$ *of orthogonal projection onto the subspace* $\sqrt{g}H^{(n)}$ *and the operator* Π^g *of orthogonal projection onto the subspace* $\sqrt{g}H$ *satisfy*

$$\Pi^{(g,n)} \to \Pi^g \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu) \text{ as } n \to \infty\,.$$

Corollary 3.7. *In the notation and under the assumptions of Proposition* 3.6*, we have*

(1) $\Psi_g \in L_1(\mathrm{Conf}(E), \mathbb{B}^{(n)})$ *for all* n, $\Psi_g \in L_1(\mathrm{Conf}(E), \mathbb{B})$*;*

(2)
$$\frac{\Psi_g \mathbb{B}^{(n)}}{\int\limits_{\mathrm{Conf}(E)} \Psi_g \, d\mathbb{B}^{(n)}} \to \frac{\Psi_g \mathbb{B}}{\int\limits_{\mathrm{Conf}(E)} \Psi_g \, d\mathbb{B}}$$

with respect to the weak topology on $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}(E))$ *as* $n \to \infty$.

Indeed, the Proposition and the Corollary are immediate from the characterization of multiplicative functionals of infinite determinantal measures given in Proposition 2.14 and Corollary2.19, the sufficient conditions of convergence of induced processes and finite-rank perturbations given in Propositions 3.3, 3.5, and the characterization of convergence with respect to the weak topology on $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}(E))$ given in Proposition 3.1.

3.4. **Convergence of approximating kernels and the proof of Proposition 1.3.** Our next aim is to show that, under certain additional assumptions, if a sequence g_n of measurable functions converges to 1, then the operators Π^{g_n} considered in Proposition 4.7 converge to Q in $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$.

Given two closed subspaces H_1, H_2 in $L_2(E,\mu)$, let $\alpha(H_1,H_2)$ be the angle between H_1 and H_2, defined as the infimum of angles between all nonzero vectors in H_1 and H_2; recall that if one of the subspaces has finite dimension, then the infimum is achieved.

Proposition 3.8. *Let* L, V, *and* E_0 *satisfy Assumption* 3, *and assume additionally that we have* $V \cap L_2(E,\mu) = 0$. *Let* $g_n : E \to (0,1]$ *be a sequence of positive measurable functions such that*

(1) *for all* $n \in \mathbb{N}$ *we have* $\sqrt{1-g_n}Q\sqrt{1-g_n} \in \mathscr{I}_1(E,\mu)$;
(2) *for all* $n \in \mathbb{N}$ *we have* $\sqrt{g_n}V \subset L_2(E,\mu)$;
(3) *there exists* $\alpha_0 > 0$ *such that for all* n *we have*
$$\alpha(\sqrt{g_n}H, \sqrt{g_n}V) \geq \alpha_0;$$
(4) *for any bounded* $B \subset E$ *we have*
$$\inf_{n\in\mathbb{N}, x\in E_0\cup B} g_n(x) > 0; \ \lim_{n\to\infty} \sup_{x\in E_0\cup B} |g_n(x) - 1| = 0.$$

Then, as $n \to \infty$, *we have*
$$\Pi^{g_n} \to Q \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E,\mu).$$

Using the second remark after Theorem 2.11, one can extend Proposition 3.8 also to nonnegative functions that admit zero values. Here we restrict ourselves to characteristic functions of the form $\chi_{E_0\cup B}$ with B bounded, in which case we have the following

Corollary 3.9. *Let B_n be an increasing sequence of bounded Borel sets exhausting $E \setminus E_0$. If there exists $\alpha_0 > 0$ such that for all n we have*

$$\alpha(\chi_{E_0 \cup B_n} H, \chi_{E_0 \cup B_n} V) \geq \alpha_0,$$

then

$$\Pi^{E_0 \cup B_n} \to Q \text{ in } \mathscr{I}_{1,\mathrm{loc}}(E, \mu).$$

Informally, Corollary 3.9 means that, as n grows, the induced processes of our determinantal measure on subsets $\mathrm{Conf}(E; E_0 \cup B_n)$ converge to the "unperturbed" determinantal point process $\mathbb{P}_Q$.

Note that Proposition 1.3 is an immediate corollary of Proposition 3.8 and Corollary 3.9. Proof of Proposition 3.8.

We start by showing that, as $n \to \infty$, we have $g_n Q \to Q$ in norm.

Indeed, take $\varepsilon > 0$ and choose a bounded set B_ε in such a way that

$$\mathrm{tr} \chi_{E \setminus (E_0 \bigcup B_\varepsilon)} Q \chi_{E \setminus (E_0 \bigcup B_\varepsilon)} < \frac{\varepsilon^2}{4}.$$

Since $g_n \to 1$ uniformly on $E_) \bigcup B_\varepsilon$, we have

$$\chi_{E_0 \bigcup B_\varepsilon} (g_n - 1) Q \to 0$$

in norm as $n \to \infty$. Furthermore, we have

$$\|\chi_{E \setminus (E_0 \bigcup B_\varepsilon)} Q\| = \|\chi_{E \setminus (E_0 \bigcup B_\varepsilon)} Q\|_{\mathscr{I}_2} < \frac{\varepsilon}{2}\,.$$

Consequently, for n sufficiently big, we have:

$$\|(g_n - 1) Q\| \leqslant \|\chi_{E_0 \bigcup B_\varepsilon} (g_n - 1) Q\| + \|\chi_{E \setminus (E_0 \bigcup B_\varepsilon)} Q\| < \varepsilon\,,$$

and, since ε is arbitrary, we have, as desired, that $g_n Q \to Q$ in norm as $n \to \infty$.

In particular, we have

$$(1 + (g_n - 1) Q)^{-1} \to 1$$

in norm as $n \to \infty$.

Now, since $g_n \to 1$ uniformly on bounded sets, for any bounded Borel subset $B \subset E$, we have

$$\chi_B \sqrt{g_n} Q \to \chi_B Q \quad \text{in} \quad \mathscr{I}_2(E, \mu)$$

as $n \to \infty$. Consequently, we have

$$\chi_B \sqrt{g_n} Q \left(1 + (g_n - 1) Q\right)^{-1} Q \sqrt{g_n} \chi_B \to \chi_B Q \chi_B$$

in $\mathscr{I}_1(E, \mu)$ as $n \to \infty$, and, since B is arbitrary, we obtain

$$Q^{g_n} \to Q \quad \text{in} \quad \mathscr{I}_{1,loc}(E, \mu)\,.$$

We now let V_n be the orthogonal complement of $\sqrt{g_n} L$ in $\sqrt{g_n} L + \sqrt{g_n} V$, and let $\tilde{P}^{(n)}$ be the operator of orthogonal projection onto V_n.

By definition, we have

$$\Pi^{g_n} = Q^{g_n} + \tilde{P}^{(n)}\,.$$

To complete the proof, it suffices to establish that, as $n \to \infty$, we have

$$\tilde{P}^{(n)} \to 0 \quad \text{in} \quad \mathscr{I}_{1,loc}(E,\mu)\,,$$

to do which, since $\tilde{P}^{(n)}$ are projections onto subspaces whose dimension does not exceed that of V, it suffices to show that for any bounded set B we have $\tilde{P}^{(n)} \to 0$ in strong operator topology as $n \to \infty$.

Since the angles between subspaces $\sqrt{g_n}L$ and $\sqrt{g_n}V$ are uniformly bounded from below, it suffices to establish the strong convergence to 0 of the operators $P^{(n)}$ of orthogonal projections onto the subspaces $\sqrt{g_n}V$.

Let, therefore, $\varphi \in L_2(E,\mu)$ be supported in a bounded Borel set B; it suffices to show that $\|P^{(n)}\varphi\| \to 0$ as $n \to \infty$. But since $V \bigcap L_2(E,\mu) = 0$, for any $\varepsilon > 0$ there exists a bounded set $B_\varepsilon \supset B$ such that for any $\psi \in V$ we have

$$\frac{\|\chi_B \psi\|}{\|\chi_{B_\varepsilon}\psi\|} < \varepsilon^2.$$

We have

$$\begin{aligned}(51)\quad \|\Pi^{B_\varepsilon}\varphi\|^2 &= \langle \varphi, \Pi^{B_\varepsilon}\varphi\rangle = \\ &= \langle \varphi, \chi_B \Pi^{B_\varepsilon}\varphi\rangle \leqslant \|\varphi\|\,\|\chi_B\Pi^{B_\varepsilon}\varphi\| \leqslant \\ &\leqslant \|\varphi\|\,\varepsilon\|\Pi^{B_\varepsilon}\varphi\| \leqslant \varepsilon\|\varphi\|\,\|\Pi^{B_\varepsilon}\varphi\| \leqslant \varepsilon\|\varphi\|^2\,.\end{aligned}$$

It follows that $\|\Pi^{B_\varepsilon}\varphi\| < \varepsilon\|\varphi\|$ and, since $g_n \to 1$ uniformly on B', also that $\|P^{(n)}\varphi\| < \varepsilon\|\varphi\|$ if n is sufficiently large. Since ε is arbitrary,

$$\|P^{(n)}\varphi\| \to 0 \text{ as } n \to \infty,$$

and the Proposition is proved completely.

4. Weak compactness of families of determinantal measures.

4.1. **Configurations and finite measures.** In a similar way as the Bessel point process of Tracy and Widom is the weak limit of its finite-dimensional approximations, the infinite determinantal measure $\tilde{\mathbb{B}}^{(s)}$, the sigma-finite analogue of the Bessel point process for the values of s smaller than -1, will be seen to be the scaling limit of its finite dimensional approximations, the infinite analogues of the Jacobi polynomial ensembles. In this section, we develop the formalism necessary for obtaining scaling limits of infinite determinantal measures. To do so, we will multiply our measures by finite densities. normalize and establish convergence of the resulting determinantal probability measures. In the Appendix, we recall the well-known result claiming that, for finite determinantal measures induced by projection operators, local trace class convergence of the operators implies weak convergence of the determinantal measures (considered as measures on the space of Radon measures on the phase space). In order to prove the vanishing of the "Gaussian parameter" and to establish convergence of finite-dimensional approximations on the Pickrell set, we will however need a finer notion of convergence of probability measures on spaces of configurations: namely, under some additional assumptions we will code configurations by finite measures and determinantal measures by measures on the space of probability measures on the phase space. We proceed to precise definitions.

Let f be a nonnegative measurable function on E, set

$$\mathrm{Conf}_f(E) = \{X : \sum_{x \in X} f(x) < \infty\},$$

and introduce a map $\sigma_f : \mathrm{Conf}_f(E) \to \mathfrak{M}_{\mathrm{fin}}(E)$ by the formula

$$\sigma_f(X) = \sum_{x \in X} f(x)\delta_x.$$

(where δ_x stands, of course, for the delta-measure at x).

Recall that the *intensity* $\xi\mathbb{P}$ of a probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is a sigma-finite measure on E defined, for a bounded Borel set $B \subset E$, by the formula

$$\xi\mathbb{P}(B) = \int\limits_{\mathrm{Conf}(E)} \#_B(X) d\mathbb{P}(X).$$

In particular, for a determinantal measure $\mathbb{P}_K$ corresponding to an operator K on $L_2(E, \mu)$ admitting a continuous kernel $K(x, y)$, the intensity is, by definition, given by the formula

$$\xi\mathbb{P}_K = K(x, x)\mu.$$

By definition, we have the following

Proposition 4.1. *Let f be a nonnegative continuous function on E, and let $\mathbb{P}$ be a probability measure on $\mathrm{Conf}(E)$. If $f \in L_1(E, \xi\mathbb{P})$, then $\mathbb{P}(\mathrm{Conf}_f(E)) = 1$.*

Under the assumptions of Proposition 4.1, the map σ_f is $\mathbb{P}$-almost surely well-defined, and the measure $(\sigma_f)_*\mathbb{P}$ is a Borel probability measure on the space $\mathfrak{M}_{\mathrm{fin}}(E)$, that is, an element of the space $\mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(E))$.

4.2. **Weak compactness and weak convergence in the space of configurations and in the space of finite measures.** We start by formulating a tightness criterion for such families of measures.

Proposition 4.2. *Let f be a nonnegative continuous function on E. Let $\{\mathbb{P}_\alpha\}$ be a family of Borel probability measures on $\mathrm{Conf}(E)$ such that*

(1) *$f \in L_1(E, \xi\mathbb{P}_\alpha)$ for all α and*

$$\sup_\alpha \int\limits_E f d\xi\mathbb{P}_\alpha < +\infty;$$

(2) *for any $\varepsilon > 0$ there exists a compact set $B_\varepsilon \subset E$ such that*

$$\sup_\alpha \int\limits_{E\setminus B_\varepsilon} f d\xi\mathbb{P}_\alpha < \varepsilon.$$

Then the family $(\sigma_f)_*\mathbb{P}_\alpha$ is tight in $\mathfrak{M}_{\text{fin}}\,(\mathfrak{M}_{\text{fin}}(E))$.

Remark. The assumptions of Proposition 4.2 can be equivalently reformulated as follows: the measures $(\sigma_f)_*\mathbb{P}_\alpha$ are all well-defined and the family $f\xi\mathbb{P}_\alpha$ is tight in $\mathfrak{M}_{\text{fin}}(E)$.

Proof of Proposition 4.2. Given $\varepsilon > 0$, our aim is to find a compact set $C \subset \mathfrak{M}_{\text{fin}}(E)$ such that $(\sigma_f)_*\mathbb{P}_\alpha(C) > 1 - \varepsilon$ for all α.

Let $\varphi : E \to \mathbb{R}$ be a bounded function. Define a measurable function $\mathrm{int}_\varphi : \mathfrak{M}_{\text{fin}}(E) \to \mathbb{R}$ by the formula

$$\mathrm{int}_\varphi(\eta) = \int_E \varphi d\eta.$$

Given a Borel subset $A \subset E$, for brevity we write $\mathrm{int}_A(\eta) = \mathrm{int}_{\chi_A}$.

The following proposition is immediate from local compactness of the space E and the weak compactness of the space of Borel probability measures on a compact metric space.

Proposition 4.3. *Let $L > 0,\ \varepsilon_n > 0,\ \lim\limits_{n\to\infty} \varepsilon_n = 0$. Let $K_n \subset E$ be compact sets such that $\bigcup\limits_{n=1}^{\infty} K_n = E$. The set*

$$\{\eta \in \mathfrak{M}_{\text{fin}}(E) : \mathrm{int}_E(\eta) \leqslant L, \mathrm{int}_{E\setminus K_n}(\eta) \leqslant \varepsilon_n \ for\ all\ n \in \mathbb{N}\}$$

is compact in the weak topology on $\mathfrak{M}_{\text{fin}}(E)$.

The Prohorov Theorem together with the Chebyshev Inequality now immediately implies

Corollary 4.4. *Let $L > 0,\ \varepsilon_n > 0,\ \lim\limits_{n\to\infty} \varepsilon_n = 0$. Let $K_n \subset E$ be compact sets such that $\bigcup\limits_{n=1}^{\infty} K_n = E$. Then the set*

$$\{\nu \in \mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(E)) : \int\limits_{\mathfrak{M}_{\mathrm{fin}}(E)} \mathrm{int}_E(\eta) d\nu(\eta) \leqslant L, \quad \int\limits_{\mathfrak{M}_{\mathrm{fin}}(E)} \mathrm{int}_{E\setminus K_n}(\eta) d\nu(\eta) \leqslant \varepsilon_n \textit{ for all } n \in \mathbb{N}\} \tag{52}$$

is compact in the weak topology on $\mathfrak{M}_{\mathrm{fin}}(E)$.

Corollary 4.4 implies Proposition 4.2. First, the total mass of the measures $f\xi\mathbb{P}_\alpha$ is uniformly bounded, which, by the Chebyshev inequality, implies, for any $\varepsilon > 0$, the existence of the constant L such that for all α we have

$$(\sigma_f)_*\mathbb{P}_\alpha(\{\eta \in \mathfrak{M}_{\mathrm{fin}}(E) : \eta(E) \leq L\}) > 1 - \varepsilon.$$

Second, tightness of the family $f\xi\mathbb{P}_\alpha$ precisely gives, for any $\varepsilon > 0$, a compact set $K_\varepsilon \subset E$ satisfying, for all α, the inequality

$$\int\limits_{\mathfrak{M}_{\mathrm{fin}}(E)} \mathrm{int}_{E\setminus K_\varepsilon}(\eta) d(\sigma_f)_*\mathbb{P}_\alpha(\eta) \leqslant \varepsilon.$$

Finally, choosing a sequence ϵ_n decaying fast enough and using Corollary 4.4, we conclude the proof of Proposition 4.2.

We now give sufficient conditions ensuring that convergence in the space of measures on the space of configurations implies convergence of corresponding measures on the space of finite measures.

Proposition 4.5. *Let f be a nonnegative continuous function on E. Let $\mathbb{P}_n, n \in \mathbb{N}$, $\mathbb{P}$ be Borel probability measures on $\mathrm{Conf}(E)$ such that*

1. *$\mathbb{P}_n \to \mathbb{P}$ with respect to the weak topology on $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}(E))$ as $n \to \infty$;*
2. *$f \in L_1(E, \xi\mathbb{P}_n)$ for all $n \in \mathbb{N}$;*
3. *the family $f\xi\mathbb{P}_n$ is a tight family of finite Borel measures on E.*

Then $\mathbb{P}(\mathrm{Conf}_f(E)) = 1$ and the measures $(\sigma_f)_*\mathbb{P}_n$ converge to $(\sigma_f)_*\mathbb{P}$ weakly in $\mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(E))$ as $n \to \infty$.

Proposition 4.5 easily follows from Proposition 4.2. First, we restrict ourselves to the open subset $\{x \in E : f(x) > 0\}$ which itself is a complete separable metric space with respect to the induced topology. Next observe that the total mass of the measures $f\xi\mathbb{P}_\alpha$ is uniformly bounded, which, by the Chebyshev inequality, implies, for any $\varepsilon > 0$, the existence of the constant L such that for all n we have

$$\mathbb{P}_n\left(\{X \in \mathrm{Conf}(E) : \sum_{x\in X} f(X) \leq L\}\right) > 1-\varepsilon.$$

Since the measures $\mathbb{P}_n$ converge to $\mathbb{P}$ weakly in $\mathfrak{M}_{\text{fin}}(\mathrm{Conf}(E))$ and the set $\{X \in \mathrm{Conf}(E) : \sum\limits_{x\in X} f(X) \leq L\}$ is closed in $\mathrm{Conf}(E)$, it follows that

$$\mathbb{P}\left(\{X \in \mathrm{Conf}(E) : \sum_{x\in X} f(X) \leq L\}\right) > 1-\varepsilon,$$

and, consequently, that $\mathbb{P}(\mathrm{Conf}_f(E)) = 1$, and the measure $(\sigma_f)_*\mathbb{P}$ is well-defined.

The family $(\sigma_f)_*\mathbb{P}_n$ is tight and must have a weak accumulation point $\mathbb{P}'$.

Using the weak convergence $\mathbb{P}_n \to \mathbb{P}$ in $\mathfrak{M}_{\text{fin}}(\mathrm{Conf}(E))$, we now show that the finite-dimensional distributions of $\mathbb{P}'$ coincide with those of $(\sigma_f)_*\mathbb{P}$. Here we use the assumption that our function f is positive and, consequently, bounded away from zero on every bounded subset of our locally compact space E.

Indeed, let $\varphi_1, \dots, \varphi_l : E \to \mathbb{R}$ be continuous functions with disjoint compact supports.

By definition, the joint distribution of the random variables $\mathrm{int}_{\varphi_1}, \dots, \mathrm{int}_{\varphi_l}$ with respect to $(\sigma_f)_*\mathbb{P}_n$ coincides with the joint distribution of the random variables $\#_{\varphi_1/f}, \dots, \#_{\varphi_l/f}$ with respect to $\mathbb{P}_n$. As $n \to \infty$, this joint distribution converges to the joint distribution of $\#_{\varphi_1/f}, \dots, \#_{\varphi_l/f}$ with respect to $\mathbb{P}$ which on the one hand, coincides with the the joint distribution of the random variables $\mathrm{int}_{\varphi_1}, \dots, \mathrm{int}_{\varphi_l}$ with respect to $(\sigma_f)_*\mathbb{P}$ and, on the other hand, also coincides with the joint distribution of the random variables $\mathrm{int}_{\varphi_1}, \dots, \mathrm{int}_{\varphi_l}$ with respect to $\mathbb{P}'$.

By Proposition 9.1, the finite-dimensional distributions determine a measure uniquely. Therefore,

$$\mathbb{P}' = (\sigma_f)_*\mathbb{P},$$

and the proof is complete.

4.3. **Applications to determinantal point processes.** Let f be a nonnegative continuous function on E. If an operator $K \in \mathscr{I}_{1,\text{loc}}(E,\mu)$ induces a determinantal measure $\mathbb{P}_K$ and satisfies $fK \in \mathscr{I}_1(E,\mu)$, then

$$\mathbb{P}_K(\mathrm{Conf}_f(E)) = 1. \tag{53}$$

If, additionally, K is assumed to be self-adjoint, then the weaker requirement $\sqrt{f}K\sqrt{f} \in \mathscr{I}_1(E,\mu)$ also implies (53).

In this special case, a sufficient condition for tightness takes the following form.

Proposition 4.6. *Let f be a bounded nonnegative continuous function on E. Let $K_\alpha \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be a family of self-adjoint positive contractions such that*

$$\sup_\alpha \mathrm{tr}\sqrt{f}K_\alpha\sqrt{f} < +\infty$$

and such that for any $\varepsilon > 0$ there exists a bounded set $B_\varepsilon \subset E$ such that

$$\sup_\alpha \mathrm{tr}\chi_{E\setminus B_\varepsilon}\sqrt{f}K_\alpha\sqrt{f}\chi_{E\setminus B_\varepsilon} < \varepsilon.$$

Then the family of measures $\left\{(\sigma_f)_*\mathbb{P}_{K_\alpha},\right\}$ is weakly precompact in $\mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(E))$.

4.4. **Induced processes corresponding to functions assuming values in $[0,1]$.** Let $g : E \to [0,1]$ be a nonnegative Borel function, and, as before, let $\Pi \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be an orthogonal projection operator with range L inducing a determinantal measure $\mathbb{P}_\Pi$ on $\mathrm{Conf}(E)$. Since the values of g do not exceed 1, the multiplicative functional Ψ_g is automatically integrable. In this particular case Proposition 9.3 of the Appendix can be reformulated as follows:

Proposition 4.7. *If $\sqrt{1-g}\Pi\sqrt{1-g} \in \mathscr{I}_1(E,\mu)$ and $||(1-g)\Pi|| < 1$, then*

(1) *Ψ_g is positive on a set of positive measure;*
(2) *the subspace $\sqrt{g}L$ is closed, and the operator Π^g of orthogonal projection onto the subspace $\sqrt{g}L$ is locally of trace class;*
(3) *we have*

$$\mathbb{P}_{\Pi^g} = \frac{\Psi_g\mathbb{P}_\Pi}{\int\limits_{\mathrm{Conf}(E)}\Psi_g\, d\mathbb{P}_\Pi}. \tag{54}$$

Remark. Since the operator $\sqrt{1-g}\Pi$ is, by assumption, Hilbert-Schmidt, and the the values of g do not exceed 1, the condition $||(1-g)\Pi|| < 1$ is equivalent to the condition $||\sqrt{1-g}\Pi|| < 1$ and both are equivalent to the nonexistence of a function $\Phi \in L$ supported on the set $\{x \in E : g(x) = 1\}$. In particular, if the function g is strictly positive, the condition is automatically verified. Proposition 4.6 now implies

Corollary 4.8. *Let f be a bounded nonnegative continuous function on E. Under the assumptions of Proposition 4.7, if*

$$\mathrm{tr}\sqrt{f}\Pi\sqrt{f} < +\infty,$$

then also

$$\mathrm{tr}\sqrt{f}\Pi^g\sqrt{f} < +\infty.$$

Proof: Equivalently, we must prove that if the operator $\sqrt{f}\Pi$ is Hilbert-Schmidt, then the operator $\sqrt{f}\Pi^g$ is also Hilbert-Schmidt. Since $\Pi^g = \sqrt{g}\Pi(1+(g-1)\Pi)^{-1}\sqrt{g}$, the statement is immediate from the fact that Hilbert-Schmidt operators form an ideal.

4.5. **Tightness for families of induced processes.** We now give a sufficient condition for the tightness of families of measures of the form Π^g for fixed g. This condition will subsequently be used for establishing convergence of determinantal measures obtained as products of infinite determinantal measures and multiplicative functionals.

Let $\Pi_\alpha \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be a family of orthogonal projection operators in $L_2(E,\mu)$. Let L_α be the range of Π_α. Let $g: E \to [0,1]$ be a Borel function such that for each α the assumptions of Proposition 4.7 are satisfied and thus the operators Π_α^g and the corresponding determinantal measures $\mathbb{P}_{\Pi_\alpha^g}$ are well-defined for all α. Furthermore, let f be a nonnegative function on E such that such that for all α we have

$$\sup_\alpha \operatorname{tr}\sqrt{f}\Pi_\alpha\sqrt{f} < +\infty \tag{55}$$

and such that for any $\varepsilon > 0$ there exists a bounded Borel set $B_\varepsilon \subset E$ such that

$$\sup_\alpha \operatorname{tr}\chi_{E\setminus B_\varepsilon}\sqrt{f}\Pi_\alpha\sqrt{f}\chi_{E\setminus B_\varepsilon} < \varepsilon. \tag{56}$$

(in other words, f is such that all the assumptions of Proposition 4.6 are satisfied for all α). It follows from Corollary 4.8 that the measures $(\sigma_f)_*\mathbb{P}_{\Pi_\alpha^g}$ are also well-defined for all α.

Sufficient conditions for tightness of this family of operators are given in the following

Proposition 4.9. *In addition to the requirements, for all α, of Proposition* 4.6 *and Proposition* 4.7*, make the assumption*

$$\inf_\alpha 1 - ||(1-g)\Pi_\alpha|| > 0. \tag{57}$$

Then the family of measures $\left\{(\sigma_f)_*\mathbb{P}_{\Pi_\alpha^g}\right\}$ is weakly precompact in $\mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(E))$.

Proof. The requirement (57) implies that the norms of the operators

$$(1+(g-1)\Pi_\alpha)^{-1}$$

are uniformly bounded in α. Recalling that $\Pi_\alpha^g = \sqrt{g}\Pi_\alpha(1+(g-1)\Pi_\alpha)^{-1}\sqrt{g}$, we obtain that (55) implies

$$\sup_\alpha \operatorname{tr}\sqrt{f}\Pi_\alpha^g\sqrt{f} < +\infty, \tag{58}$$

while (56) implies

$$\sup_{\alpha} \operatorname{tr}\chi_{E\setminus B_\varepsilon}\sqrt{f}\Pi^g_\alpha\sqrt{f}\chi_{E\setminus B_\varepsilon} < \varepsilon. \tag{59}$$

Proposition 4.9 is now immediate from Proposition 4.6.

4.6. **Tightness of families of finite-rank deformations.** We next remark that, under certain additional assumptions, tightness is preserved by taking finite-dimensional deformations of determinantal processes.

As before, we let $\Pi_\alpha \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be a family of orthogonal projection operators in $L_2(E,\mu)$. Let L_α be the range of Π_α. Let $v_{(\alpha)} \in L_2(E,\mu)$ be orthogonal to L_α, let $L^v_\alpha = L\alpha \oplus \mathbb{C}v_{(\alpha)}$, and let Π^v_α be the corresponding orthogonal projection operator. By the Macchì-Soshnikov theorem, the operator Π^v_α induces a determinantal measure $\mathbb{P}_{\Pi^v_\alpha}$ on $\mathrm{Conf}(E)$. As above, we require that all the assumptions of Proposition 4.6 be satisfied for the family Π_α. The following Corollary is immediate from Proposition 4.6.

Proposition 4.10. *Assume additionally that the family of measures $f|v^{(\alpha)}|^2\mu$ is precompact in $\mathfrak{M}_{\mathrm{fin}}(E)$. Then the family of measures $\left\{(\sigma_f)_*\mathbb{P}_{\Pi^v_\alpha},\right\}$ is weakly precompact in $\mathfrak{M}_{\mathrm{fin}}\left(\mathfrak{M}_{\mathrm{fin}}(E)\right)$.*

This proposition can be extended to perturbations of higher rank. The assumption of orthogonality of v^α to L_α is too restrictive and can be weakened to an assumption that the angle between the vector and the subspace is bounded below: indeed, in that case we can orthogonalize and apply Proposition 4.10.

We thus take $m \in \mathbb{N}$ and assume that, in addition to the family of Π_α of locally trace-class projection operators considered above, for every α we are given vectors $v^{(1)}_\alpha, \dots, v^{(m)}_\alpha$ of unit length, linearly independent and independent from L_α. Set

$$L^{v,m}_\alpha = L\alpha \oplus \mathbb{C}v^{(1)}_\alpha \oplus \mathbb{C}v^{(m)}_\alpha,$$

and let $\Pi^{v,m}_\alpha$ be the corresponding projection operator.

By the Macchì-Soshnikov theorem, the operator $\Pi^{v,m}_\alpha$ induces a determinantal measure $\mathbb{P}_{\Pi^{v,m}_\alpha}$ on $\mathrm{Conf}(E)$. As above, we require that all the assumptions of Proposition 4.6 be satisfied for the family Π_α. We let $\mathrm{angle}(v,L)$ stand for the angle between a nonzero vector v and a closed subspce L.

Proposition 4.11. *Assume additionally that*

(1) *the family of measures $f|v^{(k)}_\alpha|^2\mu$, over all α and k, is precompact in $\mathfrak{M}_{\mathrm{fin}}(E)$;*
(2) *there exists $\delta > 0$ such that for any $k = 1,\dots,m$ and all α we have*

$$\mathrm{angle}(v^{(k)}_\alpha, L_\alpha \oplus \mathbb{C}v^{(1)}_\alpha \oplus \mathbb{C}v^{(k-1)}_\alpha) \geq \delta.$$

Then the family of measures $\left\{(\sigma_f)_*\mathbb{P}_{\Pi^{v,m}_\alpha},\right\}$ is weakly precompact in $\mathfrak{M}_{\mathrm{fin}}\left(\mathfrak{M}_{\mathrm{fin}}(E)\right)$.

The proof proceeds by induction on m. For $m = 1$, it suffices to apply Proposition 4.10 to the vector obtained by taking the orthogonal projection of $v_\alpha^{(1)}$ onto the orthogonal complement of L. For the induction step, similarly, we apply Proposition 4.10 to the vector obtained by taking the orthogonal projection of $v_\alpha^{(m)}$ onto the orthogonal complement of $L_\alpha \oplus \mathbb{C}v_\alpha^{(1)} \oplus \mathbb{C}v_\alpha^{(m-1)}$). The proposition is proved completely.

4.7. **Convergence of finite-rank perturbations.** A sufficient condition for weak convergence of determinatal measures considered as elements of the space $\mathfrak{M}_{\rm fin}(\mathfrak{M}_{\rm fin}(E))$ can be formulated as follows.

Proposition 4.12. *Let f be a nonnegative continuous function on E. Let $K_n, K \in \mathscr{I}_{1,\rm loc}$ be self-adjoint positive contractions such that $K_n \to K$ in $\mathscr{I}_{1,\rm loc}(E,\mu)$ as $n \to \infty$. Assume additionally that*

$$\sqrt{f}K_n\sqrt{f} \to \sqrt{f}K\sqrt{f} \text{ in } \mathscr{I}_1(E,\mu) \tag{60}$$

as $n \to \infty$. Then

$$(\sigma_f)_*\mathbb{P}_{K_n} \to (\sigma_f)_*\mathbb{P}_K$$

weakly in $\mathfrak{M}(\mathfrak{M}_{\rm fin}(E))$ as $n \to \infty$.

Combining Proposition 4.12 with, on the one hand, Propositions 4.9, 4.11 and, on the other hand, Propositions 3.3, 3.5 and 3.6, we arrive at the following

Proposition 4.13. (1) *In the notation and under the assumptions of Proposition* 3.3*, additionally require (*60*) to hold. Then we have*

$$\sqrt{f}\tilde{\mathfrak{B}}(g,\Pi_n)\sqrt{f} \to \sqrt{f}\tilde{\mathfrak{B}}(g,\Pi)\sqrt{f}$$

in $\mathscr{I}_1(E,\mu)$, and, consequently,

$$\mathbb{P}_{\tilde{\mathfrak{B}}(g,\Pi_n)} \to \mathbb{P}_{\tilde{\mathfrak{B}}(g,\Pi)}$$

with respect to the weak topology on $\mathfrak{M}_{\rm fin}(\mathfrak{M}_{\rm fin}(E))$ as $n \to \infty$.

(2) *In the notation and under the assumptions of Proposition* 3.5*, additionally require (*60*) to hold. Then we have*

$$\sqrt{f}\widetilde{\Pi}_n\sqrt{f} \to \sqrt{f}\widetilde{\Pi}\sqrt{f} \text{ in } \mathscr{I}_1(E,\mu) \text{ as } n \to \infty,$$

and, consequently, $\mathbb{P}_{\widetilde{\Pi}_n} \to \mathbb{P}_{\widetilde{\Pi}}$ with respect to the weak topology on $\mathfrak{M}_{\rm fin}(\mathfrak{M}_{\rm fin}(E))$ as $n \to \infty$;

(3) *In the notation and under the assumptions of Proposition* 3.6*, additionally require (*60*) to hold. Then we have*

$$\sqrt{f}\Pi^{(g,n)}\sqrt{f} \to \sqrt{f}\Pi^g\sqrt{f} \;\text{ in }\; \mathscr{I}_1(E,\mu) \;\text{ as }\; n \to \infty\ .$$

and, consequently,

$$\frac{\Psi_g \mathbb{B}^{(n)}}{\displaystyle\int \Psi_g \, d\mathbb{B}^{(n)}} \to \frac{\Psi_g \mathbb{B}}{\displaystyle\int \Psi_g \, d\mathbb{B}}$$

with respect to the weak topology on $\mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}(E))$ *as* $n \to \infty$.

5. Weak convergence of rescaled radial parts of Pickrell measures .

5.1. The case $s > -1$: finite Pickrell measures.

5.1.1. *Determinantal representation of the radial parts of finite Pickrell measures.* We go back to radial parts of Pickrell measures and start with the case $s > -1$. Recall that $P_n^{(s)}$ stand for the Jacobi polynomials corresponding to the weight $(1-u)^s$ on the interval $[-1, 1]$.

We start by giving a determinantal representation for the radial part of finite Pickrell measures: in other words, we simply rewrite the formula (6) in the coordinates $\lambda_1, \ldots, \lambda_n$. Set

$$\hat{K}_n^{(s)}(\lambda_1, \lambda_2) = \frac{n(n+s)}{2n+s} \frac{1}{(1+\lambda_1)^{s/2}(1+\lambda_2)^{s/2}} \times \\ \times \frac{P_n^{(s)}\left(\frac{\lambda_1-1}{\lambda_1+1}\right) P_{n-1}^{(s)}\left(\frac{\lambda_2-1}{\lambda_2+1}\right) - P_n^{(s)}\left(\frac{\lambda_2-1}{\lambda_2+1}\right) P_{n-1}^{(s)}\left(\frac{\lambda_1-1}{\lambda_1+1}\right)}{\lambda_1 - \lambda_2}. \tag{61}$$

The kernel $\hat{K}_n^{(s)}$ is the image of the Christoffel-Darboux kernel $\tilde{K}_n^{(s)}$ (cf. (113)) under the change of variable

$$u_i = \frac{\lambda_i - 1}{\lambda_i + 1}.$$

Another representation for the kernel $\hat{K}_n^{(s)}$ is

$$\hat{K}_n^{(s)}(\lambda_1, \lambda_2) = \frac{1}{(1+\lambda_1)^{s/2+1}(1+\lambda_2)^{s/2+1}} \times \\ \times \sum_{l=0}^{n-1} (2l+s+1) P_l^{(s)}\left(\frac{\lambda_1 - 1}{\lambda_1 + 1}\right) \cdot P_l^{(s)}\left(\frac{\lambda_2 - 1}{\lambda_2 + 1}\right). \tag{62}$$

The kernel $\hat{K}_n^{(s)}$ is by definition the kernel of the operator of orthogonal projection in $L_2((0,+\infty),\mathrm{Leb})$ onto the subspace

$$\hat{L}^{(s,n)} = \mathrm{span}\left(\frac{1}{(\lambda+1)^{s/2+1}}P_l^{(s)}\left(\frac{\lambda_1-1}{\lambda_1+1}\right), l=0,\dots,n-1\right) = \\ = \mathrm{span}\left(\frac{1}{(\lambda+1)^{s/2+1}}\left(\frac{\lambda_1-1}{\lambda_1+1}\right)^l, l=0,\dots,n-1\right). \tag{63}$$

Proposition 1.17 implies the following determinantal representation the radial part of the Pickrell measure.

Proposition 5.1. *For $s>-1$, we have*

$$(\mathfrak{rad}_n)_*\mu_n^{(s)} = \frac{1}{n!}\det \hat{K}_n^{(s)}(\lambda_i,\lambda_j)\prod_{i=1}^n d\lambda_i. \tag{64}$$

5.1.2. *Scaling.* For $\beta>0$, let $\mathrm{hom}_\beta:(0,+\infty)\to(0,+\infty)$ be the homothety map that sends x to βx ; we keep the same symbol for the induced scaling transformation of $\mathrm{Conf}((0,+\infty))$.

We now give an explicit determinantal representation for the measure

$$\left(\mathrm{conf}\circ\mathrm{hom}_{n^2}\circ\mathfrak{rad}_n\right)_*\mu_n^{(s)}, \tag{65}$$

the push-forward to the space of configurations of the rescaled radial part of the Pickrell measure $\mu_n^{(s)}$.

Consider the rescaled Christoffel-Darboux kernel

$$K_n^{(s)} = n^2\tilde{K}_n^{(s)}\left(n^2\lambda_1, n^2\lambda_2\right) \tag{66}$$

of orthogonal projection onto the rescaled subspace

$$L^{(s,n)} = \mathrm{span}\left(\frac{1}{(n^2\lambda+1)^{s/2+1}}P_l^{(s)}\left(\frac{n^2\lambda_1-1}{n^2\lambda_1+1}\right)\right) = \\ = \mathrm{span}\left(\frac{1}{(n^2\lambda+1)^{s/2+1}}\left(\frac{n^2\lambda_1-1}{n^2\lambda_1+1}\right)^l, l=0,\dots,n-1\right). \tag{67}$$

The kernel $K_n^{(s)}$ induces a detrminantal process $\mathbb{P}_{K_n^{(s)}}$ on the space $\mathrm{Conf}((0,+\infty))$.

Proposition 5.2. *For $s>-1$, we have*

$$\left(\mathrm{hom}_{n^2}\circ\mathfrak{rad}_n\right)_*\mu_n^{(s)} = \frac{1}{n!}\det K_n^{(s)}(\lambda_i,\lambda_j)\prod_{i=1}^n d\lambda_i.$$

Equivalently,

$$\left(\mathrm{conf}\circ\mathrm{hom}_{n^2}\circ\mathfrak{rad}_n\right)_*\mu_n^{(s)} = \mathbb{P}_{K_n^{(s)}}.$$

5.1.3. *Scaling limit.* The scaling limit for radial parts of finite Pickrell measures is a variant of the well-known result of Tracy and Widom [43] claiming that the scaling limit of Jacobi orthogonal polynomial ensembles is the Bessel point process.

Proposition 5.3. *For any $s > -1$, as $n \to \infty$, the kernel $K_n^{(s)}$ converges to the kernel $J^{(s)}$ uniformly in the totality of variables on compact subsets of $(0, +\infty) \times (0, +\infty)$. We therefore have*

$$K_n^{(s)} \to J^{(s)} \text{ in } \mathscr{I}_{1,\mathrm{loc}}((0, +\infty), \mathrm{Leb})$$

and

$$\mathbb{P}_{K_n^{(s)}} \to \mathbb{P}_{J^{(s)}} \text{ in } \mathfrak{M}_{\mathrm{fin}}\mathrm{Conf}((0, +\infty)).$$

Proof. This is an immediate corollary of the classical Heine-Mehler asymptotics for Jacobi polynomials, see, e.g., Szegö [42].

Remark. As the Heine-Mehler asymptotics show, the uniform convergence in fact takes place on arbitrary simply connected compact subsets of $(\mathbb{C} \setminus 0) \times \mathbb{C} \setminus 0$.

5.2. The case $s \leq -1$: infinite Pickrell measures.

5.2.1. *Representation of radial parts of infinite Pickrell measures as infinite determinantal measures.* Our first aim is to show that for $s \leq -1$, the measure (34) is an infinite determinantal measure. Similarly to the definitions given in the Introduction, set

$$\hat{V}^{(s,n)} = \mathrm{span}(\frac{1}{(\lambda+1)^{s/2+1}}, \frac{1}{(\lambda+1)^{s/2+1}}\left(\frac{\lambda-1}{\lambda+1}\right), \dots, \\ \dots, \frac{1}{(\lambda+1)^{s/2+1}} P_{n-n_s}^{(s+2n_s-1)}\left(\frac{\lambda-1}{\lambda+1}\right)). \tag{68}$$

$$\hat{H}^{(s,n)} = \hat{V}^{(s,n)} \oplus \hat{L}^{(s+2n_s, n-n_s)}. \tag{69}$$

Consider now the rescaled subspaces

$$V^{(s,n)} = \mathrm{span}(\frac{1}{(n^2\lambda+1)^{s/2+1}}, \frac{1}{(n^2\lambda+1)^{s/2+1}}\left(\frac{n^2\lambda-1}{n^2\lambda+1}\right), \dots, \\ \dots, \frac{1}{(n^2\lambda+1)^{s/2+1}} P_{n-n_s}^{(s+2n_s-1)}\left(\frac{n^2\lambda-1}{n^2\lambda+1}\right)). \tag{70}$$

$$H^{(s,n)} = V^{(s,n)} \oplus L^{(s+2n_s, n-n_s)}. \tag{71}$$

Proposition 5.4. *Let $s \leq -1$, and let $R > 0$ be arbitrary. The radial part of the Pickrell measure is then an infinite determinantal measure corresponding to the subspace $H = \hat{H}^{(s,n)}$ and the subset $E_0 = (0, R)$:*

$$(\mathfrak{rad}_n)_* \mu_n^{(s)} = \mathbb{B}\left(\hat{H}^{(s,n)}, (0,R)\right).$$

For the rescaled radial part, we have

$$\mathrm{conf}_* \mathfrak{r}^{(n)}(\mu^{(s)} = (\mathrm{conf} \circ \mathrm{hom}_{n^2} \circ \mathfrak{rad}_n)_* \mu_n^{(s)} = \mathbb{B}\left(H^{(s,n)}, (0,R)\right).$$

5.3. **The modified Bessel point process as the scaling limit of the radial parts of infinite Pickrell measures: formulation of Proposition 5.5.** Denote $\mathbb{B}^{(s,n)} = \mathbb{B}\left(H^{(s,n)}, (0,R)\right)$. We now apply the formalism of the previous sections to describe the limit transition of the measures $\mathbb{B}^{(s,n)}$ to $\mathbb{B}^{(s)}$: namely, we multiply our sequence of infinite measures by a convergent multiplicative functional and establish the convergence of the resulting sequence of determinantal probability measures. It will be convenient to take $\beta > 0$ and set $g^\beta(x) = \exp(-\beta x)$, while for f it will be convenient to take the function $f(x) = \min(x, 1)$. Set, therefore,

$$L^{(n,s,\beta)} = \exp(-\beta x/2) H^{(s,n)}.$$

It is clear by definition that $L^{(n,s,\beta)}$ is a closed subspace of $L_2((0,+\infty), \mathrm{Leb})$; let $\Pi^{(n,s,\beta)}$ be the corresponding orthogonal projection operator. Recall also from (17), (18) the operator $\Pi^{(s,\beta)}$ of orthogonal projection onto the subspace $L^{(s,\beta)} = \exp(-\beta x/2) H^{(s)}$.

Proposition 5.5. (1) *For all $\beta > 0$ we have $\Psi_{g^\beta} \in L_1\left(\mathrm{Conf}(0,+\infty), \mathbb{B}^{(s)}\right)$ and, for all $n > -s+1$ we also have $\Psi_{g^\beta} \in L_1\left(\mathrm{Conf}(0,+\infty), \mathbb{B}^{(s,n)}\right)$;*

(2) *we have*

$$\frac{\Psi_{g^\beta} \mathbb{B}^{(s,n)}}{\displaystyle\int \Psi_{g^\beta}\, d\mathbb{B}^{(s,n)}} = \mathbb{P}_{\Pi^{(n,s,\beta)}};$$

$$\frac{\Psi_{g^\beta} \mathbb{B}^{(s)}}{\displaystyle\int \Psi_{g^\beta}\, d\mathbb{B}^{(s)}} = \mathbb{P}_{\Pi^{(s,\beta)}};$$

(3) *We have*

$$\Pi^{(n,s,\beta)} \to \Pi^{s,\beta} \text{ in } \mathscr{I}_{1,\mathrm{loc}}((0,+\infty), \mathrm{Leb})) \text{ as } n \to \infty .$$

and, consequently,

$$\mathbb{P}_{\Pi^{(n,s,\beta)}} \to \mathbb{P}_{\Pi^{(s,\beta)}}$$

as $n \to \infty$ weakly in $\mathfrak{M}_{\mathrm{fin}}\left(\mathrm{Conf}\big((0,+\infty)\big)\right)$;

(4) *for* $f(x) = \min(x,1)$ *we have*

$$\sqrt{f}\Pi^{(n,s,\beta)}\sqrt{f}, \sqrt{f}\Pi^{s,\beta}\sqrt{f} \in \mathscr{I}_1((0,+\infty), \mathrm{Leb});$$

$$\sqrt{f}\Pi^{(n,s,\beta)}\sqrt{f} \to \sqrt{f}\Pi^{s,\beta}\sqrt{f} \ \textit{ in } \ \mathscr{I}_1((0,+\infty), \mathrm{Leb}) \ \textit{ as } \ n \to \infty \ .$$

and, consequently,

$$(\sigma_f)_*\mathbb{P}_{\Pi^{(g,s,n)}} \to (\sigma_f)_*\mathbb{P}_{\Pi^{(g,s)}}$$

as $n \to \infty$ *weakly in* $\mathfrak{M}_{\mathrm{fin}}\left(\mathfrak{M}_{\mathrm{fin}}\big((0,+\infty)\big)\right)$.

The proof of Proposition 5.5 will occupy the remainder of this section.

5.4. **Proof of Proposition 5.5.**

5.4.1. *Proof of the first three claims.* For $s > -1$, write

$$L^{(n,s,\beta)} = \exp(-\beta x/2)L^{(s,n)}_{Jac}, \ L^{(s,\beta)} = \exp(-\beta x/2)L^{(s)}$$

and keep the notation $\Pi^{(n,s,\beta)}$, $\Pi^{(s,\beta)}$ for the corresponding orthogonal projection operators. For $s > -1$, using the Proposition 3.3 on the convergence of induced processes, we clearly have

$$\frac{\Psi_{g^\beta}\mathbb{P}_{K_n^{(s)}}}{\displaystyle\int \Psi_{g^\beta}\, d\mathbb{P}_{K_n^{(s)}}} = \mathbb{P}_{\Pi^{(n,s,\beta)}};$$

$$\frac{\Psi_{g^\beta}\mathbb{P}_{J^{(s)}}}{\displaystyle\int \Psi_{g^\beta}\, d\mathbb{P}_{J^{(s)}}} = \mathbb{P}_{\Pi^{(s,\beta)}},$$

and also

$$\Pi^{(n,s,\beta)} \to \Pi^{s,\beta} \ \text{ in } \ \mathscr{I}_{1,\mathrm{loc}}((0,+\infty), \mathrm{Leb}) \ \text{ as } \ n \to \infty \ .$$

If $x_n \to x$ as $n \to \infty$, then, of course, for any $\alpha \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n^{2\alpha}}(n^2x_n + 1)^\alpha = x^\alpha,$$

and, by the Heine-Mehler classical asymptotics, for any $\alpha > -1$, we also have

$$\lim_{n\to\infty} 1(n^2x + 1)^{\alpha/2+1}P_n^{(\alpha)}\left(\frac{n^2x_n - 1}{n^2x_n + 1}\right) = \frac{J_\alpha(2/\sqrt{x})}{\sqrt{x}}.$$

We recall the following statement on linear independence established above in Proposition 2.21. The statement below is obtained from Proposition 2.21 by the change of variables $y = 4/x$.

Proposition 5.6. . *For any* $s \leqslant -1$, *and any* $R > 0$ *the functions*

$$x^{-s/2-1}\chi_{(R,+\infty)}, \ldots, \frac{J_{s+2n_s-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}}\chi_{(R,+\infty)} \tag{72}$$

are linearly independent and, furthermore, are independent from the space $\chi_{(R,+\infty)}L^{s+2n_s}$.

Remark. Recall that Proposition 2.21 yields in fact a stronger result: if a linear combination

$$\Phi^{(\alpha)} = \alpha_0 \frac{J_{s+2n_s-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}}\chi_{(R,+\infty)} + \sum_{i=1}^{2n_s-2} \alpha_i x^{-s/2-i}\chi_{(R,+\infty)}$$

lies in L_2, then in fact, all the coefficients are zero: $\alpha_0 = \cdots = 0$. It follows, of course, that the functions

$$e^{-\beta x/2}x^{-s/2-1}, \ldots, e^{-\beta x/2}\frac{J_{s+2n_s-1}\left(\frac{2}{\sqrt{x}}\right)}{\sqrt{x}} \tag{73}$$

are also linearly independent and independent from the space L^{s+2n_s} . Th first three claims of Proposition 5.5 follow now from its abstract counterparts established in the previous subsections: the first and the second claim follow from Corollary 2.19, while the third claim, from Proposition 3.6. We proceed to the proof of the fourth and last claim of Proposition 5.5.

5.4.2. *The asymptotics* $J^{(s)}$ *at* 0 *and at* ∞. We shall need the asymptptics of the modified Bessel kernel $J^{(s)}$ at 0 and at ∞.

We start with a simple estimate for the usual Bessel kernel J_s.

Proposition 5.7. *For any* $s > -1$ *and any* $R > 0$ *we have*

$$\int_R^{+\infty} \frac{\tilde{J}_s(y,y)}{y}dy < +\infty. \tag{74}$$

Proof. Rewrite (74) in the form

$$\int_R^{+\infty} \frac{1}{y}\int_0^1 (J_s(\sqrt{ty}))^2 dtdy = \int_0^1 dt \int_{tR}^{+\infty} \frac{(J_s(\sqrt{y}))^2}{y}dy =$$

$$= \int_0^{+\infty} \min \frac{y}{R}, 1 \cdot \frac{J_s(\sqrt{y})^2}{y}dy = 1/R\int_0^R J_s(\sqrt{y}))^2 dx + \int_R^{+\infty} \frac{J_s(\sqrt{y}))^2}{y}dy.$$

It is immediate from the asymptotics of the Bessel functions at zero and at infinity that both integrals converge, and the proposition is proved. Effectuating the charge of variable $y = 4/x$, we arrive at the following

Proposition 5.8. *For any $s > -1$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_0^\delta xJ^{(s)}(x,x)dx < \varepsilon.$$

We also need the following

Proposition 5.9. *For any $R > 0$ we have*

$$\int\limits_0^R \tilde{J}_s(y,y)dy < \infty.$$

Proof. First note that

$$\int\limits_0^R \left(J_s(\sqrt{y})\right)^2 dy < +\infty$$

since for a fixed $s > -1$ and all sufficiently small $y > 0$ we have

$$\left(J_s(\sqrt{y})\right)^2 = O(y^s).$$

Now, write

$$\int\limits_0^R \tilde{J}_s(y,y)dy = \int\limits_0^1 \int\limits_0^R J_s(\sqrt{ty})dydt \le$$

$$\le (R+1)\int\limits_0^R (J_s(\sqrt{y}))^2 dy < +\infty,$$

and the proposition is proved. Making the change of variables $y = 4/x$, we obtain

Proposition 5.10. *For any $R > 0$ we have*

$$\int_R^\infty J^{(s)}(x,x)dx < \infty.$$

5.4.3. *Uniform in n asymptotics at infinity for the kernels $K^{(n,s)}$.* We turn to the uniform asymptotic at infinity for the kernels $K^{(n,s)}$ and the limit kernel $J^{(s)}$. This uniform asymptotics is needed to establish the last claim of Proposition 5.5.

Proposition 5.11. *For any $s > -1$ and any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\sup_{n\in\mathbb{N}} \int\limits_R^{+\infty} K^{(n,s)}(x,x)dx < \varepsilon, \tag{75}$$

Proof. We start by verifying the desired estimate(75) for $s > 0$. But if $s > 0$ then the classical inequalities for Borel functions and Jacobi polynomials (see e.g. Szegö [42]) imply the existence of a constant $C > 0$ such that for all $x \geqslant 1$ we have:

$$\sup_{n\in\mathbb{N}} K^{(n,s)}(x,x) < \frac{C}{x^2}.$$

The proposition for $s > 0$ is now immediate.

To consider the remaining case $s \in (-1, 0]$, we recall that the kernels $K^{(n,s)}$ are rank-one perturbations of the kernels $K^{(n-1,s+2)}$ and note the following immediate general

Proposition 5.12. *Let* $K_n, K, \check{K}_n, \check{K} \in \mathscr{I}_{1,\mathrm{loc}}((0,+\infty), \mathrm{Leb})$ *be locally trace-class projections acting in* $L_2((0,+\infty), \mathrm{Leb})$*. Assume*

(1) $K_n \to K$, $\check{K}_n \to \check{K}$ *in* $\mathscr{I}_{1,\mathrm{loc}}((0,+\infty), \mathrm{Leb})$ *as* $n \to \infty$*;*
(2) *for any* $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$\sup_{n\to\infty} \mathrm{tr}\chi_{(R,+\infty)}K_n\chi_{(R,+\infty)} < \varepsilon,\ \mathrm{tr}\chi_{(R,+\infty)}K\chi_{(R,+\infty)} < \varepsilon;$$

(3) *there exists* $R_0 > 0$ *such that*

$$\mathrm{tr}\chi_{(R_0,+\infty)}\check{K}\chi_{(R_0,+\infty)} < \varepsilon;$$

(4) *the projection operator* $\check{K}_n$ *is a rank one perturbation of* K_n*.*

Then for any $\varepsilon > 0$ *there exists* $R > 0$ *such that*

$$\sup_{n\to\infty} \mathrm{tr}\chi_{(R,+\infty)}\check{K}_n\chi_{(R,+\infty)} < \varepsilon.$$

Proposition 5.11 is now proved completely.

5.4.4. *Uniform in* n *asymptotics at zero for the kernels* $K^{(n,s)}$ *and completion of the proof of Proposition* 5.5. We next turn to the uniform asymptotics at zero for the kernels $K^{(n,s)}$ and the limit kernel $J^{(s)}$. Again, this uniform asymptotics is needed to establish the last claim of Proposition 5.5.

Proposition 5.13. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $n \in \mathbb{N}$ *we have*

$$\int_0^\delta xK^{(n,s)}(x,x)dx < \varepsilon. \tag{76}$$

Proof. Going back to the u-variable, we reformulate our proposition as follows:

Proposition 5.14. *For any $\varepsilon > 0$ there exists $R > 0$, $n_0 \in \mathbb{N}$, such that for all $n > n_0$ we have*

$$\frac{1}{n^2} \int_{-1}^{1-R/n^2} \frac{1+u}{1-u} \tilde{K}_n^{(s)}(u,u)du < \varepsilon. \tag{77}$$

First note that the function $\frac{1+u}{1-u}$ is bounded above on $[-1, 0]$, and therefore

$$\frac{1}{n^2} \int_{-1}^{0} \frac{1+u}{1-u} \tilde{K}_n^{(s)} du \le \frac{2}{n} \int_{-1}^{1} \tilde{K}_n^{(s)}(u,u)du = \frac{2}{n}.$$

We proceed to estimating

$$\frac{1}{n^2} \int_{0}^{1-R/n^2} \frac{1+u}{1-u} \tilde{K}_n^{(s)}(u,u)du$$

Fix $\kappa > 0$(the precise choice of κ will be described later).
Write

$$\tilde{K}_n^{(s)}(u,u) = \left(\sum_{l \le \kappa n} (2l+s+1) \left(P_l^{(s)}(u)\right)^2 \right) (1-u)^s + \\ + \left(\sum_{l > \kappa n} (2l+s+1) \left(P_l^{(s)}\right)^2 \right) (1-u)^s \tag{78}$$

We start by estimating

$$\frac{1}{n^2} \sum_{l \le \kappa n} \int_{1-1/l^2}^{1-R/n^2} \frac{1+u}{1-u} (2l+s+1)(P_l^{(s)})^2 (1-u)^s du \tag{79}$$

Using the trivial estimate

$$\max_{u \in [-1,1]} \left| P_l^{(s)}(u) \right| = O(l^2)$$

we arrive, for the integral (79), at the upper bound

$$const \cdot \frac{1}{n^2} \sum_{l \le \kappa n} l^{2s+1} \cdot \int_{1-\frac{1}{l^2}}^{1-\frac{c}{n^2}} (1-u)^{s-1} du \tag{80}$$

We now consider three cases: $s > 0$, $s = 0$, and $-1 < s < 0$.

The First Case. If $s > 0$, then the integral (80) is estimated above by the expression

$$const \cdot \frac{1}{n^2} \cdot \sum_{l \le \kappa n} l^{2s+1} \frac{1}{l^{2s}} \le const \cdot \kappa^2.$$

The Second Case. If $s = 0$, then the integral (80) is estimated above by the expression

$$const \cdot \frac{1}{n^2} \sum_{l \le \kappa n} l \cdot \log(\frac{n}{l}) \le const \cdot \kappa^2.$$

The Third Case. Finally, if $-1 < s < 0$, then we arrive, for the integral (80), at the upper bound

$$const \cdot \frac{1}{n^2} \left(\sum_{l \le \kappa n} l^{2s+1} \right) \cdot R^s n^{-2s} \le const \cdot R^s \kappa^{2+2s}$$

Note that in this case, the upper bound decreases as R grows. Note that in all three cases the contribution of the integral (80) can be made arbitrarily small by choosing κ sufficiently small. We next estimate

$$\frac{1}{n^2} \sum_{l \le \kappa n} \int_0^{1-\frac{1}{l^2}} \frac{1+u}{1-u} (sl+s+1) \left(P_l^{(s)(u)} \right)^2 (1-u)^s du \tag{81}$$

Here we use the estimate (7.32.5) in Szegö [42] that gives

$$\left| P_l^{(s)}(u) \right| \le const \frac{(1-u)^{-\frac{s}{2}-\frac{1}{4}}}{\sqrt{n}}$$

as long as $u \in [0, 1-\frac{1}{l^2}]$ and arrive, for the integral (81), at the upper bound

$$const \cdot \frac{1}{n^3} \sum_{l \le \kappa n} l^2 \le const \cdot \kappa^3$$

which, again, can be made arbitrarily small as soon as κ is chosen sufficiently small.

It remains to estimate the integral

$$\frac{1}{n^2} \sum_{\kappa n \le l < n} \int_0^{1-\frac{R}{n^2}} \frac{1+u}{1-u} \cdot (2l+s+1)(P_l^{(s)})^2 (1-u)^s du \tag{82}$$

Here again we use the estimate (7.32.5) in Szego [42] and note that since the ratio $\frac{l}{n}$ is bounded below, we have a uniform estimate

$$\left| P_l^{(s)} \right| \le const \cdot \frac{(1-u)^{-\frac{s}{2}-\frac{1}{4}}}{\sqrt{n}}$$

valid as long as $\kappa n \leq l \leq n$, $u \in [0, 1 - \frac{R}{n^2}]$, and in which the constant depends on κ and does not grow as R grows.

We thus arrive, for integral (82), at the upper bound

$$\frac{const}{n^3} \sum_{\kappa n \leq l < n} \int\limits_0^{1-\frac{R}{n^2}} (1-u)^{-\frac{3}{2}} du \leq \frac{const}{\sqrt{R}}$$

Now choosing κ sufficiently small as a function of ε and then R sufficiently large as a function of ε and κ, we conclude the proof of the proposition.

The fourth claim of Proposition 5.5 is now an immediate corollary of uniform estimates given in Propositions 5.11, 5.13 and the general statement given in Proposition 4.13.

Proposition 5.5 is proved completely.

6. Convergence of approximating measures on the Pickrell set and proof of Propositions 1.15, 1.16.

6.1. **Proof of Proposition 1.15.** Proposition 1.15 easily follows from what has already been established. Recall that we have a natural forgetting map $\mathrm{conf}: \Omega_P \to \mathrm{Conf}(0, +\infty)$ that assigns to $\omega = (\gamma, x)$, $x = (x_1, \dots, x_n, \dots)$, the configuration $\omega(x) = (x_1, \dots, x_n, \dots)$. By definition, the map conf is $\mathfrak{r}^{(n)}(\mu^{(s)}$-almost surely bijective. The characterization of the measure $\mathrm{conf}_* \mathfrak{r}^{(n)}(\mu^{(s)}$ as an infinite determinantal measure given by Proposition 5.4 and the first statement of Proposition 5.5 now imply Proposition 1.15. We proceed to the proof of Proposition 1.16.

6.2. **Proof of Proposition 1.16.** Recall that, by definition, we have

$$\mathrm{conf}_* \nu^{(s,n,\beta)} = \mathbb{P}_{\Pi^{(s,n,\beta)}}.$$

Recall that Proposition 5.5 implies that, for any $s \in \mathbb{R}$, $\beta > 0$, as $n \to \infty$ we have

$$\mathbb{P}_{\Pi^{(s,n,\beta)}} \to \mathbb{P}_{\Pi^{(s,\beta)}}$$

in $\mathfrak{M}_{\mathrm{fin}}(\mathrm{Conf}((0, +\infty)))$ and, furthermore, setting $f(x) = \min(x, 1)$, also the weak convergence

$$(\sigma_f)_* \mathbb{P}_{\Pi^{(s,n,\beta)}} \to (\sigma_f)_* \mathbb{P}_{\Pi^{(s,\beta)}}$$

in $\mathfrak{M}_{\mathrm{fin}}(\mathfrak{M}_{\mathrm{fin}}((0, +\infty)))$. We now need to pass from weak convergence of probability measures on the space of configurations established in Proposition 5.5 to the weak convergence of probability measures on the Pickrell set.

We have a natural map

$$\mathfrak{s}\colon \Omega_P \to \mathfrak{M}_{\mathrm{fin}}\big((0, +\infty)\big)$$

defined by the formula

$$\mathfrak{s}(\omega)=\sum_{i=1}^{\infty}\min(x_i(\omega),1)\delta_{x_i(\omega)}\,.$$

The map $\mathfrak{s}$ is bijective in restriction to the subset Ω_P^0 defined, we recall, as the subset of $\omega=(\gamma,x)\in\Omega_P$ such that $\gamma=\sum x_i(\omega)$.

Remark. The function $\min(x,1)$ is chosen only for concreteness: any other positive bounded function on $(0,+\infty)$ coinciding with x on some interval $(0,\varepsilon)$ and bounded away from zero on its complement, could have been chosen instead.

Consider the set

(83)

$$\mathfrak{s}\Omega_P=\{\eta\in\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big):\eta=\sum_{i=1}^{\infty}\min(x_i,1)\delta_{x_i}\text{ for some }x_i>0\}.$$

The set $\mathfrak{s}\Omega_P$ is clearly closed in $\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)$.

Any measure η from the set $\mathfrak{s}\Omega_P$ admits a unique representation $\eta=\mathfrak{s}\omega$ for a unique $\omega\in\Omega_P^0$.

Consequently, to any finite Borel measure $\mathbb{P}\in\mathfrak{M}_{\text{fin}}(\mathfrak{M}_{\text{fin}}((0,+\infty)))$ supported on the set $\mathfrak{s}\Omega_P$ there corresponds a unique measure $\mathfrak{p}\mathbb{P}$ on Ω_P such that

(1) $\mathfrak{s}_*\mathfrak{p}\mathbb{P}=\mathbb{P}$;
(2) $\mathfrak{p}\mathbb{P}(\Omega_P\setminus\Omega_P^0)=0$.

6.3. **Weak convergence in $\mathfrak{M}_{\text{fin}}\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)$ and weak convergence in $\mathfrak{M}_{\text{fin}}(\Omega_P)$.** The connection of the weak convergence in the space of finite measures on the space of finite measures on the half-line to weak convergence on the space of measures on the Pickrell set is now given by the following

Proposition 6.1. *Let $\nu_n,\nu\in\mathfrak{M}_{\text{fin}}\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)$ be supported on the set $\mathfrak{s}\Omega_P$ and assume that $\nu_n\to\nu$ weakly in $\mathfrak{M}_{\text{fin}}\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)$ as $n\to\infty$. Then*

$$\mathfrak{p}\nu_n\to\mathfrak{p}\nu$$

weakly in $\mathfrak{M}_{\text{fin}}(\Omega_P)$ as $n\to\infty$.

The map $\mathfrak{s}$ is, of course, not continuous, since the function

$$\omega\to\sum_{i=1}^{\infty}\min(x_i(\omega),1)$$

is not continuous on the Pickrell set.

Nonetheless, we have the following relation between tightness of measures on Ω_P and on $\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)$.

Lemma 6.2. *Let* $\mathbb{P}_\alpha \in \mathfrak{M}_{\text{fin}}(\mathfrak{M}_{\text{fin}}((0,+\infty)))$ *be a tight family of measures. Then the family* $\mathfrak{p}\mathbb{P}_\alpha$ *is also tight.*

Proof. Take $R > 0$ and consider the subset

$$\Omega_P(R) = \left\{ \omega \in \Omega_P \colon \gamma(\omega) \le R, \sum_{i=1}^{\infty} \min(x_i(\omega), 1) \leqslant R \right\}.$$

The subset $\Omega_P(R)$ is compact in Ω_P.

By definition, we have

$$\mathfrak{s}\left(\Omega_P(R)\right) \subset \{\eta : \mathfrak{M}_{\text{fin}}\left((0,+\infty)\right) : \eta((0,+\infty) \le R\}.$$

Consequently, for any $\varepsilon > 0$ one can find a sufficiently large R in such a way that

$$\left(\mathfrak{s}\right)_* \mathbb{P}_\alpha\left(\mathfrak{s}\left(\Omega_P(R)\right)\right) < \varepsilon \text{ for all } \alpha\,.$$

Since all measures $\mathbb{P}_\alpha$ are supported on Ω_P^0, it follows that

$$\mathbb{P}_\alpha\left(\Omega_P(R)\right) < \varepsilon \text{ for all } \alpha\,,$$

and the desired tightness is established.

Corollary 6.3. *Let*

$$\mathbb{P}_n \in \mathfrak{M}_{\text{fin}}\left(\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)\right), n \in \mathbb{N}, \mathbb{P} \in \mathfrak{M}_{\text{fin}}\left(\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)\right)$$

be finite Borel measures. Assume

(1) *the measures* $\mathbb{P}_n$ *are supported on the set* $\mathfrak{s}\Omega_P$ *for all* $n \in \mathbb{N}$*;*
(2) $\mathbb{P}_n \to \mathbb{P}$ *converge weakly in* $\mathfrak{M}_{\text{fin}}\left(\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)\right)$ *as* $n \to \infty$*,*

then the measure $\mathbb{P}$ *is also supported on the set* $\mathfrak{s}\Omega_P$ *and* $\mathfrak{p}\mathbb{P}_n \to \mathfrak{p}\mathbb{P}$ *weakly in* $\mathfrak{M}_{\text{fin}}\left(\Omega_P\right)$ *as* $n \to \infty$*.*

Proof. The measure $\mathbb{P}$ is of course supported on the set $\mathfrak{s}\Omega_P$, since the set $\mathfrak{s}\Omega_P$ is closed. The desired weak convergence in $\mathfrak{M}_{\text{fin}}\left(\Omega_P\right)$ is now established in three steps.

The First Step: The Family $\mathfrak{p}\mathbb{P}_n$ *is Tight .*

The family $\mathfrak{p}\mathbb{P}_n$ is tight by Lemma 6.2 and therefore admits a weak accumulation point $\mathbb{P}' \in \mathfrak{M}_{\text{fin}}\left(\Omega_P\right)$.

The Second Step: Finite-Dimensional Distributions Converge.

Let $l \in \mathbb{N}$, let $\varphi_l \colon (0,+\infty) \to \mathbb{R}$ be continuous compactly supported functions, set $\varphi_l(x) = \min(x,1)\psi_l(x)$, take $t_1,\dots,t_l \in \mathbb{R}$ and observe that, by definition, for any $\omega \in \Omega_P$ we have

$$\exp\left(i\sum_{k=1}^{l} t_k \left(\sum_{r=1}^{\infty} \varphi_k\left(x_i(\omega)\right)\right)\right) = \exp\left(i\sum_{k=1}^{l} t_k \text{int}_{\psi_k}\left(\mathfrak{s}\omega\right)\right) \tag{84}$$

and consequently

$$\int\limits_{\Omega_P} \exp\left(i\sum_{k=1}^{l} t_k \left(\sum_{r=1}^{\infty} \varphi_k\left(x_i(\omega)\right)\right)\right) d\mathbb{P}'(\omega) = \tag{85}$$

$$= \int\limits_{\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)} \exp\left(i\sum_{k=1}^{l} t_k \text{int}_{\psi_k}(\eta)\right) d\left(\mathfrak{s}\right)_* \mathbb{P}'(\eta)\,.$$

We now write

$$\int\limits_{\Omega_P} \exp\left(i\sum_{k=1}^{l} t_k \left(\sum_{r=1}^{\infty} \varphi_k\left(x_i(\omega)\right)\right)\right) d\mathbb{P}'(\omega) = \tag{86}$$

$$= \lim_{n\to\infty} \int\limits_{\Omega_P} \exp\left(i\sum_{k=1}^{l} t_k \left(\sum_{r=1}^{\infty} \varphi_k\left(x_i(\omega)\right)\right)\right) d\mathbb{P}_n(\omega)\,.$$

On the other hand, since $\mathbb{P}_n \to \mathbb{P}$ weakly in $\mathfrak{M}_{\text{fin}}\left(\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)\right)$, we have

$$\lim_{n\to\infty} \int\limits_{\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)} \exp\left(i\sum_{k=1}^{l} t_k \text{int}_{\psi_k}(\eta)\right) d\left(\mathfrak{s}\right)_* \mathbb{P}_n = \tag{87}$$

$$= \int\limits_{\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)} \exp\left(i\sum_{k=1}^{l} t_k \text{int}_{\psi_k}(\eta)\right) d\mathbb{P}\,.$$

It follows that

$$\int\limits_{\Omega_P} \exp\left(i\sum_{k=1}^{l} t_k \left(\sum_{r=1}^{\infty} \varphi_k\left(x_i(\omega)\right)\right)\right) d\mathbb{P}'(\omega) = \tag{88}$$

$$= \int\limits_{\mathfrak{M}_{\text{fin}}\left((0,+\infty)\right)} \exp\left(i\sum_{k=1}^{l} t_k \text{int}_{\psi_k}(\eta)\right) d\mathbb{P}.$$

Since integrals of functions of the form $\exp\left(i\sum\limits_{k=1}^{l} t_k \text{int}_{\psi_k}(\eta)\right)$ determine a finite borel measure on $\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)$ uniquely, we have

$$\left(\mathfrak{s}\right)_* \mathbb{P}' = \mathbb{P}\,.$$

The Third Step: The Limit Measure is Supported on Ω_P^0.
To see that $\mathbb{P}'\left(\Omega_P \setminus \Omega_P^0\right) = 0$, write

$$\int\limits_{\Omega_P} e^{-\gamma(\omega)}\, d\mathbb{P}'(\omega) = \lim_{n\to\infty} \int\limits_{\Omega_P} e^{-\gamma(\omega)}\, d\mathbb{P}_n(\omega)$$

$$\int\limits_{\Omega_P} e^{-\sum\limits_{i=1}^{\infty} x_i(\omega)}\, d\mathbb{P}'(\omega) = \int\limits_{\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)} e^{-\eta\big((0,+\infty)\big)}\, d\mathbb{P}(\eta) =$$

$$= \lim_{n\to\infty} \int\limits_{\mathfrak{M}_{\text{fin}}\big((0,+\infty)\big)} e^{-\eta\big((0,+\infty)\big)}\, d\left(\sigma_x\right)_* \mathbb{P}_n = \lim_{n\to\infty} \int\limits_{\Omega_P} e^{-\sum\limits_{i=1}^{\infty} x_i(\omega)}\, d\mathbb{P}_n\,.$$

Since for any $n \in \mathbb{N}$ we have

$$\int\limits_{\Omega_P} e^{-\gamma(\omega)}\, d\mathbb{P}_n(\omega) = \int\limits_{\Omega_P} e^{-\sum\limits_{i=1}^{\infty} x_i(\omega)}\, d\mathbb{P}_n(\omega)\,,$$

it follows that

$$\int\limits_{\Omega_P} e^{-\gamma(\omega)}\, d\mathbb{P}'(\omega) = \int\limits_{\Omega_P} e^{-\sum\limits_{i=1}^{\infty} x_i(\omega)}\, d\mathbb{P}'(\omega)\,,$$

whence the equality $\gamma(\omega) = \sum\limits_{i=1}^{\infty} x_i(\omega)$ holds $\mathbb{P}'$-almost surely, and $\mathbb{P}'\left(\Omega_P \setminus \Omega_P^0\right) = 0$.

We thus have $\mathbb{P}' = \mathfrak{p}\mathbb{P}$. The proof is complete.

7. Proof of Lemma 1.14 and Completion of the Proof of Theorem 1.11.

7.1. **Reduction of Lemma 1.14 to Lemma 7.1.** Recall that we have introduced a sequence of mappings

$$\mathfrak{r}^{(n)} : \mathrm{Mat}(n, \mathbb{C}) \to \Omega_P^0, n \in \mathbb{N}$$

that to $z \in \mathrm{Mat}(n, \mathbb{C})$ assigns the point

$$\mathfrak{r}^{(n)}(z) = \left(\frac{\mathrm{tr} z^* z}{n^2}, \frac{\lambda_1(z)}{n^2}, \ldots, \frac{\lambda_n(z)}{n^2}, 0, \ldots, 0\right),$$

where $\lambda_1(z) \geqslant \ldots \geqslant \lambda_n(z) \geqslant 0$ are the eigenvalues of the matrix z^*z, counted with multiplicities and arranged in non-increasing order. By definition, we have

$$\gamma(\mathfrak{r}^{(n)}(z)) = \frac{\mathrm{tr} z^* z}{n^2}.$$

Following Vershik [44], we now introduce on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ a sequence of averaging operators over the compact groups $U(n)\times U(n)$.

$$(\mathcal{A}_n f)(z) = \int\limits_{U(n)\times U(n)} f(u_1 z u_2^{-1}) du_1 du_2, \tag{89}$$

where du stands for the normalized Haar measure on the group $U(n)$. For any $U(\infty)\times U(\infty)$-invariant probability measure on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$, the operator $\mathcal{A}_n$ is the operator of conditional expectation with respect to the sigma-algebra of $U(n)\times U(n)$-invariant sets.

By definition, the function $(\mathcal{A}_n f)(z)$ only depends on $\mathfrak{r}^{(n)}(z)$.

Lemma 7.1. *Let $m\in\mathbb{N}$. There exists a positive Schwartz function φ on $\mathrm{Mat}(m,\mathbb{C})$ as well as a positive continuous function f on Ω_P such that for any $z\in\mathrm{Mat}(\mathbb{N},\mathbb{C})$ and any $n\geq m$ we have*

$$f(\mathfrak{r}^{(n)}(z)) \leq (\mathcal{A}_n\varphi)(z). \tag{90}$$

Remark. The function φ, initially defined on $\mathrm{Mat}(m,\mathbb{C})$, is here extended to $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ in the obvious way: the value of φ at a matrix z is set to be its value on its $m\times m$ corner.

We postpone the proof of the Lemma to the next subsection and proceed with the the proof of Lemma 1.14.

Refining the definition of the class $\mathfrak{F}$ in the introduction, take $m\in\mathbb{N}$ and let $\mathcal{F}(m)$ is the family of all Borel sigma-finite $U(\infty)\times U(\infty)$-invariant measures ν on $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ such that for any $R>0$ we have

$$\nu\left(\{z : \max_{i,j\leq m}|z_{ij}|<R\}\right) < +\infty.$$

Equivalently, the measure of a set of matrices, whose $m\times m$-corners are required to lie in a compact set, must be finite; in particular, the projections $(\pi_n^\infty)_*\nu$ are well-defined for all m. For example, if $s+m>0$, then the Pickrell measure $\mu^{(s)}$ belongs to $\mathcal{F}(m)$. Recall furthermore that, by the results of [9], [10] any measure $\nu\in\mathcal{F}(m)$ admits a unique ergodic decomposition into *finite* ergodic components: in other words, for any such ν there exists a unique Borel sigma-finite measure $\overline{\nu}$ on Ω_P such that we have

$$\nu = \int\limits_{\Omega_P} \eta_\omega d\overline{\nu}(\omega). \tag{91}$$

Since the orbit of the unitary group is of course a compact set, the measures $(\mathfrak{r}^{(n)})_*\nu$ are well-defined for $n>m$ and may be thought of as finite-dimensional approximations of the decomposing measure $\overline{\nu}$. Indeed, recall from the introduction that, if ν is finite, then the measure $\overline{\nu}$ is the weak limit of the measures $(\mathfrak{r}^{(n)})_*\nu$ as $n \to \infty$. The following proposition is a stronger and a more precise version of Lemma 1.14 from the introduction.

Proposition 7.2. *Let $m \in \mathbb{N}$, let $\nu \in \mathcal{F}(m)$, let φ and f be given by Lemma* 7.1, *and assume*

$$\varphi \in L_1(\mathrm{Mat}(\mathbb{N}, \mathbb{C}), \nu).$$

Then

(1)
$$f \in L_1(\Omega_P, (\mathfrak{r}^{(n)})_*\nu)$$
for all $n > m$;

(2)
$$f \in L_1(\Omega_P, \overline{\nu});$$

(3)
$$f(\mathfrak{r}^{(n)})_*\nu \to f\overline{\nu}$$
weakly in $\mathfrak{M}_{\mathrm{fin}}(\Omega_P)$.

Proof. *First Step: The Martingale Convergence Theorem and the Ergodic Decomposition.*

We start by formulating a pointwise version of the equality (30) from the Introduction: for any $z \in \mathrm{Mat}_{\mathrm{reg}}$ and any bounded continuous function φ on $\mathrm{Mat}(\mathbb{N}; \mathbb{C})$ we have

$$\lim_{n\to\infty} \mathcal{A}_n\varphi(z) = \int\limits_{\mathrm{Mat}(\mathbb{N};\mathbb{C})} f d\eta_{\mathfrak{r}^\infty(z)} \tag{92}$$

(here, as always, given $\omega \in \Omega_P$, the symbol η_ω stands for the ergodic probability measure corresponding to ω.) Indeed, (92) immediately follows from the definition of regular matrices, the Olshanski-Vershik characterization of the convergence of orbital measures [30] and the Reverse Martingale Convergence Theorem.

The Second Step.

Now let φ and f be given by Lemma 7.1, and assume

$$\varphi \in L_1(\mathrm{Mat}(\mathbb{N}, \mathbb{C}), \nu).$$

Lemma 7.3. *for any $\varepsilon > 0$ there exists a $U(\infty) \times U(\infty)-$ invariant set $Y_\varepsilon \subset \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ such that*

(1) $\nu(Y_\varepsilon) < +\infty$;
(2) *for all $n > m$ we have*
$$\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})\setminus Y_\varepsilon} f(\mathfrak{r}^{(n)}(z)) d\nu(z) < \varepsilon.$$

Proof. Since $\varphi \in L_1(\mathrm{Mat}(\mathbb{N},\mathbb{C}),\nu)$, we have

$$\int\limits_{\Omega_P}\left(\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})}\varphi d\eta_\omega\right)d\overline{\nu}(\omega) < +\infty.$$

Choose a Borel subset $\tilde{Y}^\varepsilon \subset \Omega_P$ in such a way that $\overline{\nu}(\tilde{Y}^\varepsilon) < +\infty$ and

$$\int\limits_{\tilde{Y}^\varepsilon}\left(\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})}\varphi d\eta_\omega\right)d\overline{\nu}(\omega) < \varepsilon.$$

The pre-image of the set $\tilde{Y}^\varepsilon$ under the map ω_∞ or, more precisely, the set

$$Y_\varepsilon = \{z \in \mathrm{Mat}_{\mathrm{reg}} : \omega_\infty(z) \in \tilde{Y}^\varepsilon\}$$

is by definition $U(\infty)\times U(\infty)-$ invariant and has all the desired properties.

The Third Step.

Let $\psi : \Omega_P \to \mathbb{R}$ be continuous and bounded. Take $\varepsilon > 0$ and the corresponding set Y_ε.

For any $z \in \mathrm{Mat}_{\mathrm{reg}}$ we have

$$\lim_{n\to\infty}\psi(\mathfrak{r}^{(n)}(z))\cdot f(\mathfrak{r}^{(n)}(z)) = \psi(\omega_\infty(z))\cdot f(\omega_\infty(z)).$$

Since $\nu(Y_\varepsilon) < \infty$, the bounded convergence theorem gives

$$\lim_{n\to\infty}\int\limits_{Y_\varepsilon}\psi(\mathfrak{r}^{(n)}(z))\cdot f(\mathfrak{r}^{(n)}(z))d\nu(z) = \\ = \int\limits_{Y_\epsilon}\psi(\omega_\infty(z))\cdot f(\omega_\infty(z))d\nu(z). \tag{93}$$

By definition of Y_ε for all $n \in \mathbb{N}$, $n > m$, we have

$$\left|\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})\setminus Y_\varepsilon}\psi(\mathfrak{r}^{(n)}(z))\cdot f(\mathfrak{r}^{(n)}(z))d\nu(z)\right| \le \varepsilon\sup_{\Omega_P}|\psi|.$$

It follows that

$$\lim_{n\to\infty}\int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})}\psi(\mathfrak{r}^{(n)}(z))\cdot f(\mathfrak{r}^{(n)}(z))d\nu(z) = \\ = \int\limits_{\mathrm{Mat}(\mathbb{N},\mathbb{C})}\psi(\omega_\infty(z))\cdot f(\omega_\infty(z))d\nu(z), \tag{94}$$

which, in turn, implies that

$$\lim_{n\to\infty}\int\limits_{\Omega_P}\psi f d(\mathfrak{r}^{(n)})_*(\nu)=\int\limits_{\Omega_P}\psi f d\overline{\nu},$$

that the weak convergence is established, and that the Lemma is proved completely.

7.2. **Proof of Lemma 7.1.** Introduce an inner product $\langle,\rangle$ on $\mathrm{Mat}(m,\mathbb{C})$ by the formula $\langle z_1, z_2\rangle = \Re\mathrm{tr}(z_1^* z_2)$. This inner product is naturally extended to a pairing between the projective limit $\mathrm{Mat}(\mathbb{N},\mathbb{C})$ and the inductive limit

$$\mathrm{Mat}_0=\bigcup_{m=1}^{\infty}\mathrm{Mat}(m,\mathbb{C}).$$

For a matrix $\zeta\in\mathrm{Mat}_0$ set

$$\Xi_\zeta(z)=\exp(i\langle\zeta,z\rangle),\ z\in\mathrm{Mat}(\mathbb{N},\mathbb{C}).$$

We start with the following simple estimate on the behaviour of the Fourier transform of orbital measures.

Lemma 7.4. *Let $m\in\mathbb{N}$. For any $\varepsilon>0$ there exists $\delta>0$ such that for any $n>m$ and $\zeta\in\mathrm{Mat}(m,\mathbb{C})$, $z\in\mathrm{Mat}(\mathbb{N},\mathbb{C})$ satisfying*

$$\mathrm{tr}(\zeta^*\zeta)\mathrm{tr}\left((\pi_n^\infty(z))^*(\pi_n^\infty(z)\right)<\delta n^2$$

we have

$$|1-\mathcal{A}_n\Xi_\zeta(z)|<\varepsilon.$$

Proof. This is a simple corollary of the power series representation of the Harish-Chandra–Itzykson–Zuber orbital integral, see e.g. [14], [15], [35]. Indeed, let $\sigma_1,\cdots,\sigma_m$ be the eigenvalues of z^*z, and let $x_1^{(n)},\cdots,x_n^{(n)}$ be the eigenvalues of $\pi_n^\infty(z)$.

The standard power series representation, see e.g. [14], [15],[35], for the Harish-Chandra–Itzykson–Zuber orbital integral gives, for any $n\in\mathbb{N}$, a representation

$$A_n\Xi(z)=1+\sum_{\lambda\in\mathbb{Y}_+}a(\lambda,n)s_\lambda\left(\sigma_1,\cdots,\sigma_m\right)\cdot s_\lambda\left(\frac{x_1^{(n)}}{n^2},\cdots,\frac{x_n^{(n)}}{n^2}\right),$$

where the summation takes place over the set $\mathbb{Y}_+$ all non-empty Young diagrams λ, s_λ stands for the Schur polynomial corresponding to the diagram λ, and the coefficients $a(\lambda,n)$ satisfy

$$\sup_{\lambda\in\mathbb{Y}_+}|a(\lambda,n)|\le 1$$

The proposition follows immediately.

Corollary 7.5. *For any $m \in \mathbb{N}$, $\varepsilon > 0$, $R > 0$, there exists a positive Schwartz function $\psi : \mathrm{Mat}(m, \mathbb{C}) \to (0, 1]$ such that for all $n > m$ we have*

$$\mathcal{A}_n \psi(\pi_m^\infty(z)) \geq 1 - \varepsilon \tag{95}$$

for all z satisfying

$$\mathrm{tr}\left((\pi_n^\infty(z))^*(\pi_n^\infty(z)\right) < Rn^2.$$

Proof. Let ψ be a Schwartz function taking values in $(0, 1]$. Assume additionally that $\psi(0) = 1$ and that the Fourier transform of ψ is supported in the ball of radius ε_0 around the origin. A Schwartz function satisfying all these requirements is constructed without difficulty. By Lemma 7.4, if ε_0 is small enough as a function of m, ε, R, then the inequality (95) holds for all $n > m$.Corollary 7.5 is proved completely.

We now conclude the proof of Lemma 7.1.

Take a sequence $R_n \to \infty$, and let ψ_n be the corresponding sequence of Schwartz functions given by Corollary 7.5. Take positive numbers t_n decaying fast enough so that the function

$$\varphi = \sum_{n=1}^{\infty} t_n \psi_n$$

is Schwartz.

Let $\tilde{f}$ be a positive continuous function on $(0, +\infty)$ such that for any n, if $t \leq R_n$, then $\tilde{f}(t) < t_n/2$. For $\omega \in \Omega_P$, $\omega = (\gamma, x)$, set

$$f(\omega) = \tilde{f}(\gamma(\omega)).$$

The function f is by definition positive and continuous. By Corollary 7.5, the functions φ and f satisfy all requirements of Lemma 7.1, which, therefore, is proved completely.

7.3. **Completion of the proof of Theorem 1.11.**

Lemma 7.6. *Let E be a locally compact complete metric space. Let $\mathbb{B}_n, \mathbb{B}$ be sigma-finite measures on E, let $\mathbb{P}$ be a probability measure on E, and let f, g be positive bounded continuous functions on E. Assume that for all $n \in \mathbb{N}$ we have*

$$g \in L_1(E, \mathbb{B}_n)$$

and that, as $n \to \infty$, we have

(1)

$$f\mathbb{B}_n \to f\mathbb{B}$$

weakly in $\mathfrak{M}_{\mathrm{fin}}(E)$;

(2)
$$\frac{g\mathbb{B}_n}{\int\limits_E g d\mathbb{B}_n} \to \mathbb{P}$$

weakly in $\mathfrak{M}_{\mathrm{fin}}(E)$.

Then
$$g \in L_1(E, \mathbb{B})$$
and
$$\mathbb{P} = \frac{g\mathbb{B}}{\int\limits_E g d\mathbb{B}}$$

Proof. Let φ be a nonnegative bounded continuous function on E. On the one hand, as $n \to \infty$, we have
$$\int\limits_E \varphi f g d\mathbb{B}_n \to \int\limits_E \varphi f g d\mathbb{B},$$
and, on the other hand, we have
$$\frac{\int\limits_E \varphi f g d\mathbb{B}_n}{\int\limits_E g d\mathbb{B}_n} \to \int\limits_E \varphi f d\mathbb{P}. \tag{96}$$
Choosing $\varphi = 1$, we obtain that
$$\lim_{n\to\infty} \int\limits_E g d\mathbb{B}_n = \frac{\int\limits_E f g d\mathbb{B}}{\int\limits_E f d\mathbb{P}} > 0;$$
the sequence $\int\limits_E g d\mathbb{B}_n$ is thus bounded away both from zero and infinity. Furthermore, for arbitrary bounded continuous positive φ we have
$$\lim_{n\to\infty} \int\limits_E g d\mathbb{B}_n = \frac{\int\limits_E \varphi f g d\mathbb{B}}{\int\limits_E \varphi f d\mathbb{P}}. \tag{97}$$

Now take $R > 0$ and $\varphi(x) = \min(1/f(x), R)$. Letting R tend to ∞, we obtain

$$\lim_{n\to\infty} \int\limits_E g d\mathbb{B}_n = \int\limits_E g d\mathbb{B}. \tag{98}$$

Substituting (98) back into (96), we arrive at the equality

$$\int\limits_E \varphi f d\mathbb{P} = \frac{\int\limits_E \varphi f g d\mathbb{B}}{\int\limits_E g d\mathbb{B}}.$$

Note that here, as in (96), the function φ may be an arbitrary nonnegative continuous function on E. In particular, taking a compactly supported function ψ on E and setting $\varphi = \psi/f$, we obtain

$$\int\limits_E \psi d\mathbb{P} = \frac{\int\limits_E \psi g d\mathbb{B}}{\int\limits_E g d\mathbb{B}}.$$

Since this equality is true for any compactly supported fnction ψ on E, we conclude that

$$\mathbb{P} = \frac{g\mathbb{B}}{\int\limits_E g d\mathbb{B}},$$

and the Lemma is proved completely.

Combining Lemma 7.6 with Lemma 1.14 and Proposition 1.16, we conclude the proof of Theorem 1.11.

Theorem 1.11 is proved completely.

7.4. **Proof of Proposition 1.4.** In view of Proposition 1.10 and Theorem 1.11, it suffices to prove the singularity of the ergodic decomposition measures $\overline{\mu}^{(s_1)}$, $\overline{\mu}^{(s_2)}$. Since, by Proposition 1.9, the measures $\mu^{(s_1)}$, $\mu^{(s_2)}$ are mutually singular, there exists a set $D \subset \mathrm{Mat}(\mathbb{N}, \mathbb{C})$ such that

$$\mu^{(s_1)}(D) = 0, \mu^{(s_2)}(\mathrm{Mat}(\mathbb{N}, \mathbb{C}) \setminus D) = 0.$$

Introduce the set

$$\tilde{D} = \{z \in \mathrm{Mat}(\mathbb{N}, \mathbb{C}) : \lim_{n\to\infty} \mathcal{A}_n \chi_D(z) = 1\}.$$

By definition, the set $\tilde{D}$ is $U(\infty)\times U(\infty)$-invariant, and we have

$$\mu^{(s_1)}(\tilde{D})=0, \mu^{(s_2)}(\mathrm{Mat}(\mathbb{N},\mathbb{C})\setminus\tilde{D})=0.$$

Introduce now the set $\overline{D}\subset\Omega_P$ by the formula

$$\overline{D}=\{\omega\in\Omega_P:\eta_\omega(\tilde{D})=1\}.$$

We clearly have

$$\overline{\mu}^{(s_1)}(\overline{D})=0, \overline{\mu}^{(s_2)}(\Omega_P\setminus\overline{D})=0.$$

Proposition 1.4 is proved completely.

8. Appendix A. The Jacobi Orthogonal Polynomial Ensemble.

8.1. **Jacobi polynomials.** Let $\alpha,\beta>-1$, and let $P_n^{(\alpha,\beta)}$ be the standard Jacobi orthogonal polynomials, namely, polynomials on the unit interval $[-1,1]$ orthogonal with weight

$$(1-u)^\alpha(1+u)^\beta$$

and normalized by the condition

$$P_n^{(\alpha,\beta)}(1)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Recall that the leading term $k_n^{(\alpha,\beta)}$ of $P_n^{(\alpha,\beta)}$ is given (see e.g. (4.21.6) in Szegö [42]) by the formula

$$k_n^{(\alpha,\beta)}=\frac{\Gamma(2n+\alpha+\beta+1)}{2^n\cdot\Gamma(n+1)\cdot\Gamma(n+\alpha+\beta+1)}$$

while for the square of the norm we have

$$h_n^{(\alpha,\beta)}=\int\limits_{-1}^{1}\left(P_n^{(\alpha,\beta)}(u)\right)^2\cdot(1-u)^\alpha(1+u)^\beta\,du=$$
$$=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}. \tag{99}$$

Denote by $\tilde{K}_n^{(\alpha,\beta)}(u_1,u_2)$ the n-th Christoffel-Darboux kernel of the Jacobi orthogonal polynomial ensemble:

(100)
$$\tilde{K}_n^{(\alpha,\beta)}(u_1,u_2)=\sum_{l=0}^{n-1}\frac{P_l^{(\alpha,\beta)}(u_1)\cdot P_l^{(\alpha,\beta)}(u_2)}{h_l^{(\alpha,\beta)}}(1-u_1)^{\alpha/2}(1+u_1)^{\beta/2}(1-u_2)^{\alpha/2}(1+u_2)^{\beta/2}.$$

The Christoffel-Darboux formula gives an equivalent representation for the kernel $\tilde{K}_n^{(\alpha,\beta)}$:

$$(101)\quad \tilde{K}_n^{(\alpha,\beta)}(u_1,u_2) = \\ = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta}\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha)\Gamma(n+\beta)}\cdot(1-u_1)^{\alpha/2}(1+u_1)^{\beta/2}(1-u_2)^{\alpha/2}(1+u_2)^{\beta/2}\times \\ \times\frac{P_n^{(\alpha,\beta)}(u_1)P_{n-1}^{(\alpha,\beta)}(u_2)-P_n^{(\alpha,\beta)}(u_2)P_{n-1}^{(\alpha,\beta)}(u_1)}{u_1-u_2}.$$

8.2. **The recurrence relation between Jacobi polynomials.** We have the following recurrence relation between the Christoffel-Darboux kernels $\tilde{K}_{n+1}^{(\alpha,\beta)}$ and $\tilde{K}_n^{(\alpha+2,\beta)}$.

Proposition 8.1. *For any $\alpha,\beta>-1$ we have*

$$(102)\quad \tilde{K}_{n+1}^{(\alpha,\beta)}(u_1,u_2) = \\ = \frac{\alpha+1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)\Gamma(n+\alpha+1)}P_n^{(\alpha+1,\beta)}(u_1)(1-u_1)^{\alpha/2}(1+u_1)^{\beta/2}\times \\ \times P_n^{(\alpha+1,\beta)}(u_2)(1-u_2)^{\alpha/2}(1+u_2)^{\beta/2}+ \\ +\tilde{K}_n^{(\alpha+2,\beta)}(u_1,u_2).$$

Remark. The recurrence relation (102) can of course be taken to the scaling limit to yield a similar recurrence relation for Bessel kernels: the Bessel kernel with parameter s is thus a rank one perturbation of the Bessel kernel with parameter $s+2$. This is also easily esablished directly: using the recurrence relation

$$(103)\qquad J_{s+1}(x)=\frac{2s}{x}J_s(x)-J_{s-1}(x)$$

for Bessel functions, one immediately obtains the desired recurrence relation

$$(104)\qquad \tilde{J}_s(x,y)=\tilde{J}_{s+2}(x,y)+\frac{s+1}{\sqrt{xy}}J_{s+1}(\sqrt{x})J_{s+1}(\sqrt{y})$$

for the Bessel kernels.

Proof of Proposition 8.1. The routine calculation is included for completeness. We use standard recurrence relations for Jacobi polynomials. First, we use the relation

$$(n+\frac{\alpha+\beta}{2}+1)(u-1)P_n^{(\alpha+1,\beta)}(u)=(n+1)P_{n+1}^{(\alpha,\beta)}(u)-(n+\alpha+1)P_n^{(\alpha,\beta)}(u)$$

to arrive at the equality

$$\frac{P_{n+1}^{(\alpha,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-P_{n+1}^{(\alpha,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}= \tag{105}$$
$$=\frac{2n+\alpha+\beta+2}{2(n+1)}\frac{(u_1-1)P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-(u_2-1)P_n^{(\alpha+1,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}.$$

We next apply the relation

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(u)=(n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(u)-(n+\beta)P_{n-1}^{(\alpha+1,\beta)}(u)$$

to arrive at the equality

$$\frac{(u_1-1)P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-(u_2-1)P_n^{(\alpha+1,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}= \tag{106}$$
$$=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha+1,\beta)}(u_2)+$$
$$+\frac{n+\beta}{2n+\alpha+\beta+1}\frac{(1-u_1)P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+1,\beta)}(u_2)-(1-u_2)P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+1,\beta)}(u_1)}{u_1-u_2}.$$

Using next the recurrence relation

$$(n+\frac{\alpha+\beta+1}{2})(1-u)P_{n-1}^{(\alpha+2,\beta)}(u)=(n+\alpha+1)P_{n-1}^{(\alpha+1,\beta)}(u)-nP_n^{(\alpha+1,\beta)}(u),$$

we arrive at the equality

$$\frac{(1-u_1)P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+1,\beta)}(u_2)-(1-u_2)P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+1,\beta)}(u_1)}{u_1-u_2}= \tag{107}$$
$$=-\frac{n}{n+\alpha+1}P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha+1,\beta)}(u_2)+$$
$$+\frac{2n+\alpha+\beta+1}{2(n+\alpha+1)}(1-u_1)(1-u_2)\frac{P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}.$$

Combining (106) and (107), we obtain

(108)
$$\frac{(u_1-1)P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha,\beta)}(u_2)-(u_2-1)P_n^{(\alpha+1,\beta)}(u_2)P_n^{(\alpha,\beta)}(u_1)}{u_1-u_2}=$$
$$=\frac{(\alpha+1)(2n+\alpha+\beta+1)}{(n+\alpha+1)(2n+\alpha+\beta+1)}P_n^{(\alpha+1,\beta)}(u_1)P_n^{(\alpha+1,\beta)}(u_2)+$$
$$+\frac{n+\beta}{2(n+\alpha+1)}(1-u_1)(1-u_2)\frac{P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}.$$

Using the recurrence relation

$$(2n+\alpha+\beta+2)P_n^{(\alpha+1,\beta)}(u)=(n+\alpha+\beta+2)P_n^{(\alpha+2,\beta)}(u)-(n+\beta)P_{n-1}^{(\alpha+2,\beta)}(u),$$

we now arrive at the relation

$$\frac{P_n^{(\alpha+1,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+1,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}= \tag{109}$$
$$=\frac{n+\alpha+\beta+2}{2n+\alpha+\beta+2}\frac{P_n^{(\alpha+2,\beta)}(u_1)P_{n-1}^{(\alpha+2,\beta)}(u_2)-P_n^{(\alpha+2,\beta)}(u_2)P_{n-1}^{(\alpha+2,\beta)}(u_1)}{u_1-u_2}.$$

Combining (105), (108), (109) and recalling the definition (101) of Christoffel-Darboux kernels, we conclude the proof of Proposition 8.1.

As above, given a finite family of functions $f_1,\dots,f_N$ on the unit interval or on the real line, we let $\mathrm{span}(f_1,\dots,f_N)$ stand for the vector space these functions span. For $\alpha,\beta\in\mathbb{R}$ introduce the subspace

$$L_{Jac}^{(\alpha,\beta,n)}=\mathrm{span}((1-u)^{\alpha/2}(1+u)^{\beta/2},(1-u)^{\alpha/2}(1+u)^{\beta/2}u,\dots,$$
$$\dots,(1-u)^{\alpha/2}(1+u)^{\beta/2}u^{n-1}). \tag{110}$$

For $\alpha,\beta>-1$, Proposition 8.1 yields the following orthogonal direct-sum decomposition

$$L_{Jac}^{(\alpha,\beta,n)}=\mathbb{C}P_n^{(\alpha+1,\beta)}\oplus L_{Jac}^{(\alpha+2,\beta,n-1)}. \tag{111}$$

Though the corresponding spaces are no longer subspaces in L_2, the relation (111) is still valid for all $\alpha\in(-2,-1]$; in reformulating it, it is, however, more convenient for us to shift α by 2.

Proposition 8.2. *For all $\alpha>0$, $\beta>-1$, $n\in\mathbb{N}$ we have*

$$L_{Jac}^{(\alpha-2,\beta,n)}=\mathbb{C}P_n^{(\alpha-1,\beta)}\oplus L_{Jac}^{(\alpha,\beta,n-1)}.$$

Proof. Let $Q_n^{(\alpha,\beta)}$ be the function of the second kind corresponding to the Jacobi polynomial $P_n^{(\alpha,\beta)}$. By Szegö, [42], formula (4.62.19), for any $u\in(-1,1)$, $v>1$ we have

$$\sum_{l=0}^{n} \frac{(2l+\alpha+\beta+1)}{2^{\alpha+\beta+1}} \frac{\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}{\Gamma(l+\alpha+1)\Gamma(l+\beta+1)} P_l^{(\alpha)}(u) Q_l^{(\alpha)}(v) = \tag{112}$$

$$= \frac{1}{2}\frac{(v-1)^{-\alpha}(v+1)^{-\beta}}{(v-u)} +$$

$$+ \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2} \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \frac{P_{n+1}^{(\alpha,\beta)}(u)Q_n^{(\alpha,\beta)}(v) - Q_{n+1}^{(\alpha,\beta)}(v)P_n^{(\alpha,\beta)}(u)}{v-u}.$$

Take the limit $v \to 1$, and recall from Szegö [42], formula (4.62.5), the following asymptotic expansion as $v \to 1$ for the Jacobi function of the second kind

$$Q_n^{(\alpha)}(v) \sim \frac{2^{\alpha-1}\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}(v-1)^{-\alpha}.$$

Recalling the recurrence formula (22.7.19)in [1]:

$$P_{n+1}^{(\alpha-1,\beta)}(u) = (n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)} - (n+\beta+1)P_n^{(\alpha,\beta)}(u)$$

we arrive at the relation

$$\frac{1}{1-u} + \frac{\Gamma(\alpha)\Gamma(n+2)}{\Gamma(n+\alpha+1)} P_{n+1}^{(\alpha-1,\beta)} \in L_{Jac}^{(\alpha,\beta,n)},$$

which immediately implies Proposition 8.2.

Now take $s > -1$ and, for brevity, write $P_n^{(s)} = P_n^{(\alpha,\beta)}$. The leading term $k_n^{(s)}$ of $P_n^{(s)}$ is given by the formula

$$k_n^{(s)} = \frac{\Gamma(2n+s+1)}{2^n \cdot n! \cdot \Gamma(n+s+1)}$$

while for the square of the norm we have

$$h_n^{(s)} = \int_{-1}^{1} \left(P_n^{(s)}(u)\right)^2 \cdot (1-u)^s \, du = \frac{2^{s+1}}{2n+s+1}.$$

Denote by $\tilde{K}_n^{(s)}(u_1,u_2)$ the corresponding n-th Christoffel-Darboux kernel :

$$\tilde{K}_n^{(s)}(u_1,u_2) = \sum_{l=1}^{n-1} \frac{P_l^{(s)}(u_1) \cdot P_l^{(s)}(u_2)}{h_l^{(s)}} (1-u_1)^{s/2}(1-u_2)^{s/2}. \tag{113}$$

The Christoffel-Darboux formula gives an equivalent representation for the kernel $\tilde{K}_n^{(s)}$:

(114)

$$\tilde{K}_n^{(s)}(u_1,u_2) = \frac{n(n+s)}{2^s(2n+s)} \cdot (1-u_1)^{s/2} \cdot (1-u_2)^{s/2} \cdot \frac{P_n^{(s)}(u_1)P_{n-1}^{(s)}(u_2) - P_n^{(s)}(u_2)P_{n-1}^{(s)}(u_1)}{u_1-u_2}$$

8.3. **The Bessel kernel.** Consider the half-line $(0,+\infty)$ endowed with the standard Lebesgue measure Leb. Take $s > -1$ and consider the standard Bessel kernel

$$\tilde{J}_s(y_1,y_2) = \frac{\sqrt{y_1}J_{s+1}(\sqrt{y_1})J_s(\sqrt{y_2}) - \sqrt{y_2}J_{s+1}(\sqrt{y_2})J_s(\sqrt{y_1})}{2(y_1-y_2)} \tag{115}$$

(see, e.g., page 295 in Tracy and Widom [43]).

An alternative integral representation for the kernel $\tilde{J}_s$ has the form

$$\tilde{J}_s(y_1,y_2) = \frac{1}{4}\int\limits_0^1 J_s(\sqrt{ty_1})J_s(\sqrt{ty_2})dt \tag{116}$$

(see, e.g., formula (2.2) on page 295 in Tracy and Widom [43]).

As (116) shows, the kernel $\tilde{J}_s$ induces on $L_2((0,+\infty),\mathrm{Leb})$ the operator of orthogonal projection onto the subspace of functions whose Hankel transform is supported in $[0,1]$ (see [43]).

Proposition 8.3. *For any $s > -1$, as $n \to \infty$, the kernel $\tilde{K}_n^{(s)}$ converges to the kernel $\tilde{J}_s$ uniformly in the totality of variables on compact subsets of $(0,+\infty)\times(0,+\infty)$.*

Proof. This is an immediate corollary of the classical Heine-Mehler asymptotics for Jacobi orthogonal polynomials, see e.g. Chapter 8 in Szegö [42]. Note that the uniform convergence in fact takes place on arbitrary simply connected compact subsets of $(\mathbb{C}\setminus 0)\times\mathbb{C}\setminus 0$.

9. Appendix B. Spaces of configurations and determinantal point processes.

9.1. **Spaces of configurations.** Let E be a locally compact complete metric space.

A configuration X on E is a collection of points, called *particles* considered without regard to order; the main assumption is that perticles not accumulate anywhere in E, or, equivalently that a bounded subset of E contain only finitely many particles of a configuration.

To a configuration X assign the Radon measure

$$\sum_{x\in X}\delta_x$$

where the summation takes place over all particles of X. Conversely, any purely atomic Radon measure on E is given by a configuration. The space $\mathrm{Conf}(E)$ of configuration on E is thus identified with a closed subset of integer-valued Radon measures on E in the space of all Radon measures on E. This identification endows $\mathrm{Conf}(E)$ with the structure of a complete separable metric space, which, however, is not locally compact.

The Borel structure on $\mathrm{Conf}(E)$ can be equivalently defined as follows. For a bounded Boral subset $B \subset E$, introduce a function

$$\#_B \ : \ \mathrm{Conf}(E) \to \mathbb{R}$$

that to a configuration X assigns the number of its particles that lie in B. The family of functions $\#_B$ over alll bounded Borel subsets $B\ subsetE$ determines the Borel structure on $\mathrm{Conf}(E)$; in particular, to define a probability measure on $\mathrm{Conf}(E)$ it is necessary and sufficient to define the joint distributions of the random variables $\#_{B_1}, \cdots, \#_{B_k}$ over all finite collections of disjoint bounded Borel subsets $B_1, \cdots, B_k \subset E$.

9.2. **Weak topology on the space of probability measures on the space of configurations.** The space $\mathrm{Conf}(E)$ is endowed with a natural structure of a complete separable metric space, and the space $\mathfrak{M}_{\text{fin}}(\mathrm{Conf}(E))$ of finite Borel measures on the space of configurations is consequently also a complete separable metric space with respect to the weak topology.

Let $\varphi : E \to \mathbb{R}$ be a compactly supported continuous function. Define a measurable function $\#_\varphi : \mathrm{Conf}(E) \to \mathbb{R}$ by the formula

$$\#_\varphi(X) = \sum_{x \in X} \varphi(x).$$

For a bounded Borel subset $B \subset E$, of course we have $\#_A = \#_{\chi_A}$.

Since the Borel sigma-algebra on $\mathrm{Conf}(E)$ coincides with the sigma-algebra generated by the integer-valued random variables $\#_B$ over all bounded Borel subsets $B \subset E$, it also coincides with the sigma-algebra generated by the random variables $\#_\varphi$ over all compactly supported continuous functions $\varphi : E \to \mathbb{R}$. Consequently, we have the following

Proposition 9.1. *A Borel probability measure $\mathbb{P} \in \mathfrak{M}_{\text{fin}}(\mathrm{Conf}(E))$ is uniquely determined by the joint distributions of all finite collections*

$$\#_{\varphi_1}, \#_{\varphi_2}, \ldots, \#_{\varphi_l}$$

over all continuous functions $\varphi_1, \ldots, \varphi_l : E \to \mathbb{R}$ with disjoint compact supports.

The weak topology on $\mathfrak{M}_{\text{fin}}(\mathrm{Conf}(E))$ admits the following characterization in terms of the said finite-dimensional distributions(see Theorem

11.1.VII in vol.2 of [13]). Let $\mathbb{P}_n, n \in \mathbb{N}$ and $\mathbb{P}$ be Borel probability measures on $\mathrm{Conf}(E)$. Then the measures $\mathbb{P}_n$ converge to $\mathbb{P}$ weakly as $n \to \infty$ if and only if for any finite collection $\varphi_1, \dots, \varphi_l$ of continuous functions with disjoint compact supports the joint distributions of the random variables $\#_{\varphi_1}, \dots, \#_{\varphi_l}$ with respect to $\mathbb{P}_n$ converge, as $n \to \infty$, to the joint distribution of $\#_{\varphi_1}, \dots, \#_{\varphi_l}$ with respect to $\mathbb{P}$; convergence of joint distributions being understood according to the weak topology on the space of Borel probability measures on $\mathbb{R}^l$.

9.3. **Spaces of locally trace class operators.** Let μ be a sigma-finite Borel measure on E.

Let $\mathscr{I}_1(E,\mu)$ be the ideal of trace class operators $\widetilde{K}\colon L_2(E,\mu) \to L_2(E,\mu)$ (see volume 1 of [36] for the precise definition); the symbol $||\widetilde{K}||_{\mathscr{I}_1}$ will stand for the $\mathscr{I}_1$-norm of the operator $\widetilde{K}$. Let $\mathscr{I}_2(E,\mu)$ be the ideal of Hilbert-Schmidt operators $\widetilde{K}\colon L_2(E,\mu) \to L_2(E,\mu)$; the symbol $||\widetilde{K}||_{\mathscr{I}_2}$ will stand for the $\mathscr{I}_2$-norm of the operator $\widetilde{K}$.

Let $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be the space of operators $K\colon L_2(E,\mu) \to L_2(E,\mu)$ such that for any bounded Borel subset $B \subset E$ we have

$$\chi_B K \chi_B \in \mathscr{I}_1(E,\mu).$$

Again, we endow the space $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ with a countable family of seminorms

$$||\chi_B K \chi_B||_{\mathscr{I}_1} \tag{117}$$

where, as before, B runs through an exhausting family B_n of bounded sets.

9.4. **Determinantal Point Processes.** A Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ is called *determinantal* if there exists an operator $K \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ such that for any bounded measurable function g, for which $g-1$ is supported in a bounded set B, we have

$$\mathbb{E}_{\mathbb{P}} \Psi_g = \det\Big(1 + (g-1)K\chi_B\Big). \tag{118}$$

The Fredholm determinant in (118) is well-defined since $K \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$. The equation (118) determines the measure $\mathbb{P}$ uniquely. For any pairwise disjoint bounded Borel sets $B_1, \dots, B_l \subset E$ and any $z_1, \dots, z_l \in \mathbb{C}$ from (118) we have $\mathbb{E}_{\mathbb{P}} z_1^{\#_{B_1}} \cdots z_l^{\#_{B_l}} = \det\left(1 + \sum\limits_{j=1}^{l} (z_j - 1)\chi_{B_j} K \chi_{\sqcup_i B_i}\right)$.

For further results and background on determinantal point processes, see e.g. [4], [18], [22], [23], [24], [37], [38], [39], [41].

If K belongs to $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$, then, throughout the paper, we denote the corresponding determinantal measure by $\mathbb{P}_K$. Note that $\mathbb{P}_K$ is uniquely defined by K, but different operators may yield the same measure. By the

Macchì—Soshnikov theorem [25], [41], any Hermitian positive contraction that belongs to the class $\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ defines a determinantal point process.

9.5. **Change of variables.** Let $F : E \to E$ be a homeomorphism. The homeomorphism F induces a homeomorphism of the space $\mathrm{Conf}(E)$, for which, slightly abusing notation, we keep the same symbol: given $X \in \mathrm{Conf}(E)$, the particles of the configuration $F(X)$ have the form $F(x)$ over all $x \in X$. Now, as before, let μ be a sigma-finite measure on E, and let $\mathbb{P}_K$ be the determinantal measure induced by an operator $K \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$. Let the operator F_*K be defined by the formula $F_*K(f) = K(f \circ F)$.

Assume now that the measures $F_*\mu$ and μ are equivalent, and consider the operator

$$K^F = \sqrt{\frac{dF_*\mu}{d\mu}} F_*K \sqrt{\frac{dF_*\mu}{d\mu}}.$$

Note that if K is self-adjoint, then so is K^F. If K is given by the kernel $K(x,y)$, then K^F is given by the kernel

$$K^F(x,y) = \sqrt{\frac{dF_*\mu}{d\mu}(x)} K(F^{-1}x, F^{-1}y) \sqrt{\frac{dF_*\mu}{d\mu}(y)}.$$

Directly from the definitions we now have the following

Proposition 9.2. *The action of the homeomorphism F on the determinantal measure $\mathbb{P}_K$ is given by the formula*

$$F_*\mathbb{P}_K = \mathbb{P}_{K^F}.$$

Note that if K is the operator of orthogonal projection onto the closed subspace $L \subset L_2(E,\mu)$, then, by definition, the operator K^F is the operator of orthogonal projection onto the closed subspace

$$\{\varphi \circ F^{-1}(x) \sqrt{\frac{dF_*\mu}{d\mu}(x)}\} \subset L_2(E,\mu).$$

9.6. **Multiplicative functionals on spaces of configurations.** Let g be a non-negative measurable function on E, and introduce the *multiplicative functional* $\Psi_g : \mathrm{Conf}(E) \to \mathbb{R}$ by the formula

$$\Psi_g(X) = \prod_{x \in X} g(x). \tag{119}$$

If the infinite product $\prod\limits_{x \in X} g(x)$ absolutely converges to 0 or to ∞, then we set, respectively, $\Psi_g(X) = 0$ or $\Psi_g(X) = \infty$. If the product in the right-hand side fails to converge absolutely, then the multiplicative functional is not defined.

9.7. **Multiplicative functionals of determinantal point processes.** At the centre of the construction of infinite determinantal measures lie the results of [11], [12] that can informally be summarized as follows: a determinantal measure times a multiplicative functional is again a determinantal measure. In other words, if $\mathbb{P}_K$ is a determinantal measure on $\mathrm{Conf}(E)$ induced by the operator K on $L_2(E,\mu)$, then, under certain additional assumptions, it is shown in [11], [12] that the measure $\Psi_g\mathbb{P}_K$ after normalization yields a determinantal point process.

As before, let g be a non-negative measurable function on E. If the operator $1+(g-1)K$ is invertible, then we set

$$\mathfrak{B}(g,K)=gK(1+(g-1)K)^{-1}, \qquad \widetilde{\mathfrak{B}}(g,K)=\sqrt{g}K(1+(g-1)K)^{-1}\sqrt{g}.$$

By definition, $\mathfrak{B}(g,K),\widetilde{\mathfrak{B}}(g,K)\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ since $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$, and, if K is self-adjoint, then so is $\widetilde{\mathfrak{B}}(g,K)$.

We now recall a few propositions from [12].

Proposition 9.3. *Let $K\in\mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ be a self-adjoint positive contraction, and let $\mathbb{P}_K$ be the corresponding determinantal measure on $\mathrm{Conf}(E)$. Let g be a nonnegative bounded measurable function on E such that*

$$\sqrt{g-1}K\sqrt{g-1}\in\mathscr{I}_1(E,\mu) \tag{120}$$

and that the operator $1+(g-1)K$ is invertible. Then

(1) *we have $\Psi_g\in L_1(\mathrm{Conf}(E),\mathbb{P}_K)$ and*
$$\int\Psi_g\,d\mathbb{P}_K=\det\Big(1+\sqrt{g-1}K\sqrt{g-1}\Big)>0;$$

(2) *the operators $\mathfrak{B}(g,K),\widetilde{\mathfrak{B}}(g,K)$ induce on $\mathrm{Conf}(E)$ a determinantal measure $\mathbb{P}_{\mathfrak{B}(g,K)}=\mathbb{P}_{\widetilde{\mathfrak{B}}(g,K)}$ satisfying*

$$\mathbb{P}_{\mathfrak{B}(g,K)}=\frac{\Psi_g\mathbb{P}_K}{\displaystyle\int\limits_{\mathrm{Conf}(E)}\Psi_g\,d\mathbb{P}_K}. \tag{121}$$

Remark. Since (120) holds and K is self-adjoint, the operator $1+(g-1)K$ is invertible if and only if the operator $1+\sqrt{g-1}K\sqrt{g-1}$ is invertible.

If Q is a projection operator, then the operator $\tilde{\mathfrak{B}}(g,Q)$ admits the following description.

Proposition 9.4. *Let $L\subset L_2(E,\mu)$ be a closed subspace, and let Q be the operator of orthogonal projection onto L. Let g be a bounded measurable function such that the operator $1+(g-1)Q$ is invertible. Then the operator*

$\tilde{\mathfrak{B}}(g,Q)$ *is the operator of orthogonal projection onto the closure of the subspace* $\sqrt{g}L$.

We now consider the particular case when g is a characteristic function of a Borel subset. In much the same way as before, if $E' \subset E$ is a Borel subset such that the subspace $\chi_{E'}L$ is closed (recall that a sufficient condition for that is provided in Proposition 2.18), then we set $Q^{E'}$ to be the operator of orthogonal projection onto the closed subspace $\chi_{E'}L$.

Proposition 9.3 now yields the following

Corollary 9.5. *Let* $Q \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu)$ *be the operator of orthogonal projection onto a closed subspace* $L \in L_2(E,\mu)$. *Let* $E' \subset E$ *be a Borel subset such that* $\chi_{E'}Q\chi_{E'} \in \mathscr{I}_1(E,\mu)$. *Then*

$$\mathbb{P}_Q(\mathrm{Conf}(E,E')) = \det(1-\chi_{E\setminus E'}Q\chi_{E\setminus E'}).$$

Assume, additionally, that for any function $\varphi \in L$, *the equality* $\chi_{E'}\varphi = 0$ *implies* $\varphi = 0$. *Then the subspace* $\chi_{E'}L$ *is closed, and we have*

$$\mathbb{P}_Q(\mathrm{Conf}(E,E')) > 0,\ Q^{E'} \in \mathscr{I}_{1,\mathrm{loc}}(E,\mu),$$

and

$$\frac{\mathbb{P}_Q|_{\mathrm{Conf}(E,E')}}{\mathbb{P}_Q(\mathrm{Conf}(E,E'))} = \mathbb{P}_{Q^{E'}}. \tag{122}$$

The induced measure of a determinantal measure onto the subset of configurations all whose particles lie in E' is thus again a determinantal measure. In the case of a discrete phase space, related induced processes were considered by Lyons [22] and by Borodin and Rains [7].

We now give a sufficient condition for the almost sure *positivity* of a multiplicative functional.

Proposition 9.6. *If*

$$\mu\left(\{x \in E\colon g(x) = 0\}\right) = 0$$

and

$$\sqrt{|g-1|}K\sqrt{|g-1|} \in \mathscr{I}_1(E,\mu),$$

then

$$0 < \Psi_g(X) < +\infty$$

for $\mathbb{P}_K$*-almost all* $X \in \mathrm{Conf}(E)$.

Proof. Our assumptions imply that for $\mathbb{P}_K$-almost all $X \in \mathrm{Conf}(E)$ we have

$$\sum_{x\in X} |g(x)-1| < +\infty\,,$$

which, in turn, is sufficient for absolute convergence of the infinite product $\prod_{x\in X} g(x)$ to a finite non-zero limit.

We also formulate a version of Proposition 9.3 in the special case when the function g does not assume values less than 1. In this case the multiplicative functional Ψ_g is automatically non-zero, and we have

Proposition 9.7. *Let $\Pi \in \mathscr{I}_{1,loc}(E,\mu)$ be the operator of orthogonal projection onto a closed subspace H, let g be a bounded Borel function on E satisfying $g(x) \geq 1$ for all $x \in E$, and assume*

$$\sqrt{g-1}\Pi\sqrt{g-1} \in \mathscr{I}_1(E,\mu)\,.$$

Then:

(1) $\Psi_g \in L_1(\mathrm{Conf}(E), \mathbb{P}_\Pi)$, *and*

$$\int \Psi_g \, d\mathbb{P}_\Pi = \det\left(1+\sqrt{g-1}\Pi\sqrt{g-1}\right);$$

(2) *we have*

$$\frac{\Psi_g \mathbb{P}_\Pi}{\displaystyle\int \Psi_g \, d\mathbb{P}_\Pi} = \mathbb{P}_{\Pi^g},$$

where Π^g is the operator of orthogonal projection onto the subspace $\sqrt{g}H$.

10. Appendix C. Construction of Pickrell Measures and proof of Propositions 1.8 and 1.9.

10.1. **Proof of Proposition 1.8.** First we recall that the Pickrell measures are naturally defined on the space of *rectangular* $m \times n$-matrices.

Let $\mathrm{Mat}(m\times n,\mathbb{C})$ be the space of $m\times n$ matrices with complex entries:

$$\mathrm{Mat}(m\times n,\mathbb{C}) = \{z=(z_{ij}),\ i=1,\dots,m; j=1,\dots,n\}$$

Denote dz the Lebesgue measure on $\mathrm{Mat}(m\times n,\mathbb{C})$.

Take $s\in\mathbb{R}$. Let m_0, n_0 be such that $m_0+s>0, n_0+s>0$. Following Pickrell, take $m>m_0$, $n>n_0$ and introduce a measure $\mu^{(s)}_{m,n}$ on $\mathrm{Mat}(m\times n,\mathbb{C})$ by the formula

$$\mu^{(s)}_{m,n} = const^{(s)}_{m,n}\cdot\det(1+z^*z)^{-m-n-s}\times dm(Z), \tag{123}$$

where

$$const^{(s)}_{m,n} = \pi^{-mn}\cdot\prod_{l=m_0}^{m}\frac{\Gamma(l+s)}{\Gamma(n+l+s)}. \tag{124}$$

For $m_1\leq m$, $n_1\leq n$, let

$$\pi^{m,n}_{m_1,n_1} : \mathrm{Mat}(m\times n,\mathbb{C}) \to \mathrm{Mat}(m_1\times n_1,\mathbb{C})$$

be the natural projection map.

Proposition 10.1. *Let* $m, n \in \mathbb{N}$ *be such that* $s > -m-1$. *Then for any* $\tilde{z} \in \mathrm{Mat}(n, \mathbb{C})$ *we have*

$$\int\limits_{(\pi^{m+1,n}_{m,n})^{-1}(\tilde{z})} \det(1+z^*z)^{-m-n-1-s} dz = \pi^n \frac{\Gamma(m+1+s)}{\Gamma(n+m+1+s)} \det(1+\tilde{z}^*\tilde{z})^{-m-n-s}. \tag{125}$$

Proposition 1.8 is an immediate corollary of Proposition 10.1.

Proof of Proposition 10.1. As we noted in the Introduction, the following computation goes back to the classical work of Hua Loo Keng [19]. Take $z \in Mat\big((m+1) \times n, \mathbb{C}\big)$. Multiplying, if necessary, by a unitary matrix on the left and on the right, represent the matrix $\pi^{m+1,n}_{m,n} z = \widetilde{z}$ in diagonal form, with positive real entries on the diagonal: $\widetilde{z}_{ii} = u_i > 0$, $i = 1, \ldots, n$, $\widetilde{z}_{ij} = 0$ for $i \neq j$.

Here we set $u_i = 0$ for $i > \min(n, m)$. Denote $\xi_i = z_{m+1,i}$, $i = 1, \ldots, n$. Write

$$\det\left(1+z^*z\right)^{-m-1-n-s} = \prod_{i=1}^{m} (1+u_i^2)^{-m-1-n-s} \times \left(1 + \xi^*\xi - \sum_{i=1}^{n} \frac{|\xi_i|^2 u_i^2}{1+u_i^2}\right)^{-m-1-n-s}.$$

We have

$$1 + \xi^* \xi - \sum_{i=1}^{n} \frac{|\xi_i|^2 u_i^2}{1+u_i^2} = 1 + \sum_{i=1}^{n} \frac{|\xi_i|^2}{1+u_i^2}.$$

Integrating in ξ, we find

$$\int \left(1 + \sum_{i=1}^{n} \frac{|\xi_i|^2}{1+u_i^2}\right)^{-m-1-n-s} d\xi = \prod_{i=1}^{m} (1+u_i^2) \frac{\pi^n}{\Gamma(n)} \int\limits_0^{+\infty} r^{n-1}(1+r)^{-m-1-n-s} dr,$$

where

$$r = r^{(m+1,n)}(z) = \sum_{i=1}^{n} \frac{|\xi_i|^2}{1+u_i^2}. \tag{126}$$

Recalling the Euler integral

$$\int\limits_0^{+\infty} r^{n-1}(1+r)^{-m-1-n-s} dr = \frac{\Gamma(n)\cdot\Gamma(m+1+s)}{\Gamma(n+1+m+s)}, \tag{127}$$

we arrive at the desired conclusion. Furthermore, introduce a map

$$\widetilde{\pi}^{m+1,n}_{m,n} \colon Mat\big((m+1)\times n, \mathbb{C}\big) \longrightarrow Mat\big(m \times n, \mathbb{C}\big) \times \mathbb{R}_+$$

by the formula

$$\widetilde{\pi}^{m+1,n}_{m,n}(z) = \left(\pi^{m+1,n}_{m,n}(z)\,,\, r^{(m+1,n)}(z)\right)\,,$$

where $r^{(m+1,n)}(z)$ is given by the formula (126).

Let $P^{(m,n,s)}$ be a probability measure on $\mathbb{R}_+$ given by the formula:

$$dP^{(m,n,s)}(r) = \frac{\Gamma(n+m+s)}{\Gamma(n)\cdot\Gamma(m+s)}\, r^{n-1}(1-r)^{-m-n-s}dr\,.$$

The measure $P^{(m,n,s)}$ is well-defined as soon as $m+s>0$.

Corollary 10.2. *For any $m,n\in\mathbb{N}$ and $s>-m-1$, we have*

$$\left(\widetilde{\pi}^{m+1,n}_{m,n}\right)_* \mu^{(s)}_{m+1,n} = \mu^{(s)}_{m,n}\times P^{(m+1,n,s)}\,.$$

Indeed, this is precisely what was shown by our computation.

Removing a column is similar to removing a row:

$$\left(\pi^{m,n+1}_{m,n}(z)\right)^t = \pi^{m+1,n}_{m,n}\left(z^t\right)\,.$$

Write $\widetilde{r}^{(m,n+1)}(z) = r^{(n+1,m)}(z^t)$. Introduce a map

$$\widetilde{\pi}^{m,n+1}_{m,n}\colon Mat\big(m\times(n+1),\mathbb{C}\big)\longrightarrow Mat\big(m\times n,\mathbb{C}\big)\times\mathbb{R}_+$$

by the formula

$$\widetilde{\pi}^{m,n+1}_{m,n}(z) = \left(\pi^{m,n+1}_{m,n}(z)\,,\, \widetilde{r}^{(m,n+1)}(z)\right)\,.$$

Corollary 10.3. *For any $m,n\in\mathbb{N}$ and $s>-m-1$, we have*

$$\left(\widetilde{\pi}^{m,n+1}_{m,n}\right)_* \mu^{(s)}_{m,n+1} = \mu^{(s)}_{m,n}\times P^{(n+1,m,s)}\,.$$

Now take n such that $n+s>0$ and introduce a map

$$\widetilde{\pi}_n\colon Mat\big(\mathbb{N}\times\mathbb{N},\mathbb{C}\big)\longrightarrow Mat\big(n\times n,\mathbb{C}\big)$$

by the formula

$$\widetilde{\pi}_n(z) = \left(\pi^{\infty,\infty}_{n,n}(z)\,,\, r^{(n+1,n)},\, \widetilde{r}^{(n+1,n+1)},\, r^{(n+2,n+1)},\, \widetilde{r}^{(n+2,n+2)},\ldots\right)\,.$$

We can now reformulate the result of our computations as follows:

Proposition 10.4. *If $n+s>0$, then we have*

$$(\widetilde{\pi}_n)_*\,\mu^{(s)} = \mu^{(s)}_{m,n}\times\prod_{l=0}^{\infty}\left(P^{(n+l+1,n+l,s)}\times P^{(n+l+1,n+l+1,s)}\right)\,. \tag{128}$$

10.2. **Proof of Proposition 1.9.** Using Kakutani's theorem, we now conclude the proof of Proposition 1.9. Take n large enough so that $n+s>1$, $n+s'>1$ and compute the Hellinger integral

$$Hel\,(n,s,s') = \mathbb{E}\left(\sqrt{(P^{(n,n-1,s)}\times P^{(n,n,s)})\cdot(P^{(n,n-1,s')}\times P^{(n,n,s')})}\right) =$$

$$= \sqrt{\frac{\Gamma(2n-1+s)}{\Gamma(n-1)\Gamma(n+s)}\cdot\frac{\Gamma(2n-1+s')}{\Gamma(n-1)\Gamma(n+s')}\cdot\frac{\Gamma(2n+s)}{\Gamma(n)\Gamma(n+s)}\cdot\frac{\Gamma(2n+s')}{\Gamma(n)\Gamma(n+s')}}\times$$

$$\times\int\limits_0^\infty r^{n-1}(1+r)^{-2n-1-\frac{s+s'}{2}}dr\cdot\int\limits_0^\infty r^{n-1}(1+r)^{-2n-\frac{s+s'}{2}}dr =$$

$$= \frac{\sqrt{\Gamma(2n-1+s)\cdot\Gamma(2n-1+s')}}{\Gamma(2n-1+\frac{s+s'}{2})}\cdot\frac{\sqrt{\Gamma(2n+s)\cdot\Gamma(2n+s')}}{\Gamma(2n+\frac{s+s'}{2})}\cdot\frac{\left(\Gamma(n+\frac{s+s'}{2})\right)^2}{\Gamma(n+s)\cdot\Gamma(n+s')}\,.$$

We now recall a classical asymptotics: as $t\to\infty$, we have

$$\frac{\Gamma(t+a_1)\cdot\Gamma(t+a_2)}{\left(\Gamma(t+\frac{a_1+a_2}{2})\right)^2} = 1+\frac{(a_1+a_2)^2}{4t}+O\left(\frac{1}{t^2}\right).$$

It follows that

$$Hel\,(n,s,s') = 1-\frac{(s+s')^2}{8\,n}+O\left(\frac{1}{n^2}\right)\,,$$

whence, by the Kakutani's theorem combined with (128), the Pickrell measures $\mu^{(s)}$ and $\mu^{(s')}$, finite or infinite, are mutually singular if $s\neq s'$.

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